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Periodic solutions of a generalized system of piecewise linear difference equations

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Abstract

In this article we consider the global behavior of the following system of piecewise linear difference equations: $x_{n+1} = |x_n| - y_n - b$ and $y_{n+1} = x_n + |y_n|$ where b is any positive real number and the initial condition (x_0, y_0) is an element of R^2 . By mathematical induction and direct computations we show that the solution to the system is eventually one of two particular prime period 3 solutions or the unique equilibrium solution.

Keywords: difference equations; periodic solutions; equilibrium solution

1 Introduction

Recently there has been a surge of interest in systems of both rational and piecewise linear difference equations due to their practical applications in evolutionary biology, neural networks, and ecology [1–3]. There has been particular interest in the global behavior of systems of piecewise linear difference equations [1, 4–7].

We continue the trend by generalizing one of our recent papers [8]. In this paper we consider the behavior of the generalized system of piecewise linear difference equations,

$$\begin{cases} x_{n+1} = |x_n| - y_n - b, \\ y_{n+1} = x_n + |y_n|, \end{cases} \quad n = 0, 1, \dots,$$

where the parameter $b \in (0, \infty)$ and the initial condition $(x_0, y_0) \in R^2$. This system originated from a larger project at the University of Rhode Island which involves the following family of systems:

$$\begin{cases} x_{n+1} = |x_n| + ay_n + b, \\ y_{n+1} = x_n + c|y_n| + d, \end{cases} \quad n = 0, 1, \dots,$$

where the parameters $a, b, c, d \in \{-1, 0, 1\}$ and the initial condition $(x_0, y_0) \in R^2$. Each of the 81 systems in this family is designated a number N , where $N = 27(a + 1) + 9(b + 1) + 3(c + 1) + (d + 1) + 1$. The system considered in this paper is a generalized form of system number 8. Related work has been given recently in [8, 9].

Interest in the area began in 1984 when Devaney published his famous paper introducing the Gingerbreadman map:

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1, \\ y_{n+1} = x_n, \end{cases} \quad n = 0, 1, \dots,$$

with initial the condition $(-0.1, 0)$. See [2, 3]. The Gingerbreadman map was Devaney's response to the 1978 generalized Lozi equation. The Lozi equation,

$$\begin{cases} x_{n+1} = -a|x_n| + y_n + 1, \\ y_{n+1} = bx_n, \end{cases} \quad n = 0, 1, \dots,$$

with parameters $a, b \in \mathbb{R}$ and the initial condition $(x_0, y_0) \in \mathbb{R}^2$, had been used to examine an attractor that was observed by Lorenz in the Hénon map, a non-linear system of difference equations,

$$\begin{cases} x_{n+1} = -ax_n^2 + y_n + 1, \\ y_{n+1} = bx_n, \end{cases} \quad n = 0, 1, \dots,$$

with parameters $a, b \in \mathbb{R}$ and the initial condition $(x_0, y_0) \in \mathbb{R}^2$ that modeled weather patterns [10–12]. We believe that our family of piecewise linear difference equations are the prototypes for more elaborate piecewise difference equations that, in many cases as in the Lozi equation, exhibit complicated behavior.

2 Preliminaries

The following definitions [2] are used in this paper. A *difference equation of the first order* is an equation of the form

$$x_{n+1} = f(x_n), \quad n = 0, 1, \dots, \quad (1)$$

where f is a continuous function which maps some set J into J . The set J is usually an interval of real numbers, or a union of intervals, but it may even be a discrete set such as the set of integers.

A *solution* of (1) is a sequence $\{x_n\}_{n=0}^{\infty}$ which satisfies (1) for all $n \geq 0$. If we prescribe a set of *initial conditions*

$$x_0 \in J,$$

then

$$x_1 = f(x_0),$$

$$x_2 = f(x_1),$$

...

and so the solution $\{x_n\}_{n=0}^{\infty}$ of (1) exists for all $n \geq 0$ and is uniquely determined by the initial conditions.

A solution of (1) which is constant for all $n \geq 0$ is called an *equilibrium solution* of (1).

A solution $\{x_n\}_{n=0}^{\infty}$ of (1) is called *periodic with period p* (or a *period p cycle*) if there exists an integer $p \geq 1$ such that

$$x_{n+p} = x_n \quad \text{for all } n \geq 0. \quad (2)$$