

# Some Basic Properties of $\Gamma$ -left-right Derivation in $\Gamma$ -CI-Algebras

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**ABSTRACT**— The purpose of this paper is to introduce the notion of a  $\Gamma$ -CI-algebras, we study  $\Gamma$ -filters,  $\Gamma$ -self-distributives,  $\Gamma$ -transitives, upper sets and  $\Gamma$ -left-right  $f$ -derivations in  $\Gamma$ -CI-algebras. Some characterizations of  $\Gamma$ -filters, upper sets and  $\Gamma$ -left-right  $f$ -derivations are obtained. Moreover, we investigate relationships between  $\Gamma$ -filters,  $\Gamma$ -self-distributives and  $\Gamma$ -transitives in  $\Gamma$ -CI-algebras.

**Keywords**—  $\Gamma$ -CI-algebra,  $\Gamma$ -filter,  $\Gamma$ -self-distributive,  $\Gamma$ -transitive,  $\Gamma$ -self-distributive

## 1. INTRODUCTION

In what follows, let  $X$  denote a CI-algebra unless otherwise specified. By a CI-algebra we mean an algebra  $(X; *, 1)$  of type  $(2, 0)$  with a single binary operation “ $*$ ” that satisfies the following identities: for any  $x, y, z \in X$ ,

1.  $x * x = 1$ ,
2.  $1 * x = x$ ,
3.  $x * (y * z) = y * (x * z)$

In, 2006, H. S. Kim and Y. H. Kim defined a BE-algebra [8]. Biao Long Meng, defined notion of CI-algebra as a generation of a BE-algebra BE-algebras and CI-algebras are studied in detail by some researchers (A. Borumand Saeid and A. Rezaei, 2012) and (B. Piekart and A. Andrzej Walendziak, 2011) some fundamental properties of CI-algebra are discussed.

In this paper is to introduce the notion of a  $\Gamma$ -CI-algebras, we study  $\Gamma$ -filters,  $\Gamma$ -self-distributives,  $\Gamma$ -transitives and upper sets in  $\Gamma$ -CI-algebras. Some characterizations of  $\Gamma$ -filters and upper sets are obtained. Moreover, we investigate relationships between  $\Gamma$ -filters,  $\Gamma$ -self-distributives and  $\Gamma$ -transitives in  $\Gamma$ -CI-algebras.

## 2. BASIC PROPERTIES

In this section we refer to [12] for some elementary aspects and quote few definitions and examples which are essential to step up this study. For more detail we refer to the papers in the references.

**Definition 2.1.** [12] Let  $X$  and  $\Gamma$  be any nonempty sets. If there exists a mapping  $X \times \Gamma \times X \rightarrow X$  written as  $(x, \gamma, y)$  by  $x\gamma y$ , then  $(\Gamma, X; 1)$  is called a  $\Gamma$ -CI-algebra if

1.  $x\gamma x = 1$  for all  $x \in X$  and  $\gamma \in \Gamma$ ,
2.  $1\gamma x = x$  for all  $x \in X$  and  $\gamma \in \Gamma$ ,
3.  $x\gamma(y\beta z) = y\gamma(x\beta z)$

for all  $x, y, z \in X$  and  $\gamma, \beta \in \Gamma$ .

A  $\Gamma$ -CI-algebra  $(\Gamma, X; 1)$  is said to be  $\Gamma$ -self-distributive if  $x\gamma(y\beta z) = (x\gamma y)\beta(x\gamma z)$  for all  $x, y, z \in X$  and  $\gamma, \beta \in \Gamma$ .

**Example 2.2.** [12] Let  $(\Gamma, X; 1)$  be an arbitrary CI-algebra and  $\Gamma$  any nonempty set. Define a mapping  $X \times \Gamma \times X \rightarrow X$ , by  $x\gamma y = xy$  for all  $x, y \in X$  and  $\gamma \in \Gamma$ . It is easy to see that  $X$  is a  $\Gamma$ -CI-algebra. Indeed,

$$\begin{aligned} 1. \quad x\gamma x &= xx = 1, \\ 2. \quad 1\gamma x &= 1x = x, \\ 3. \quad x\gamma 1 &= x1 = x, \\ 4. \quad x\gamma(y\beta z) &= x(yz) = y(xz) \end{aligned}$$

for all  $x, y, z \in X$  and  $\gamma, \beta \in \Gamma$ . Thus every CI-algebra implies a  $\Gamma$ -CI-algebra.

**Example 2.3.** [12] Let  $X = \{1, a, b\}$  in which “ $\cdot$ ” is defined by

| $\cdot$ | 1 | a | b |
|---------|---|---|---|
| 1       | 1 | b | a |
| a       | a | 1 | b |
| b       | b | a | 1 |

Then  $X$  is not a CI-algebra. Let  $\Gamma \neq \emptyset$ . Define a mapping  $X \times \Gamma \times X \rightarrow X$  by  $x\gamma y = y \cdot x$  for all  $x, y \in X$  and  $\gamma \in \Gamma$ . Then  $X$  is a  $\Gamma$ -CI-algebra. Indeed,

$$\begin{aligned} 1. \quad x\gamma x &= (x \cdot x) \cdot (x \cdot x) \\ &= 1 \cdot 1 \\ &= 1, \\ 2. \quad 1\gamma x &= x \cdot 1 \\ &= x, \\ 3. \quad 1\gamma x &= x \cdot 1 \\ &= x, \\ 4. \quad x\gamma(y\beta z) &= (y\beta z) \cdot x \\ &= (z \cdot y) \cdot x \\ &= (z \cdot x) \cdot y \\ &= (x\beta z) \cdot y \\ &= y\gamma(x\beta z) \end{aligned}$$

for all  $x, y, z \in X$  and  $\gamma, \beta \in \Gamma$ . Therefore  $X$  is a  $\Gamma$ -CI-algebra.

**Lemma 2.4.** [12] Let  $X$  be a  $\Gamma$ -CI-algebra. Then  $x\gamma y = x\beta y$  for any  $x, y \in X$  and  $\gamma, \beta \in \Gamma$ .

**Proposition 2.5.** [12] Any  $\Gamma$ -CI-algebra  $X$  satisfies for any  $x, y \in X$  and  $\gamma, \beta, \alpha \in \Gamma$ ,  $y\gamma[(y\alpha x)\beta x] = 1$

**Proposition 2.6.** [12] Any  $\Gamma$ -CI-algebra  $X$  satisfies for any  $x, y \in X$  and  $\gamma, \beta \in \Gamma$ ,  $(x\gamma 1)\beta(y\gamma 1) = (x\gamma y)\beta 1$ .

**Proposition 2.7.** [12] Let  $X$  be a  $\Gamma$ -CI-algebra. If  $x\gamma(x\beta y) = x\beta y$  for any  $x, y \in X$  and  $\gamma, \beta \in \Gamma$ , then  $x\gamma 1 = 1$ .

### 3. $\Gamma$ -FILTERS IN $\Gamma$ -CI-ALGEBRAS

In this section, we study  $\Gamma$ -filters,  $\Gamma$ -self-distributives,  $\Gamma$ -transitives,  $\Gamma$ -self-distributive and upper sets in  $\Gamma$ -CI-algebras.

**Definition 3.1.** Let  $X$  be a  $\Gamma$ -CI-algebra and  $F$  a non-empty subset of  $X$ . Then  $F$  is said to be a  $\Gamma$ -filter of  $X$  if

1.  $1 \in F$ ,
2. If  $x \in F$  and  $x\gamma y \in F$ , then  $y \in F$ .

**Proposition 3.2.** If  $F_i$ , for all  $i \in I$  are  $\Gamma$ -filters of  $X$ , then  $\bigcap_{i \in I} F_i$  is a  $\Gamma$ -filter of  $\Gamma$ -CI-algebra  $X$ .

**Proof.** Straightforward.

**Proposition 3.3.** Let  $F$  be a subset of  $\Gamma$ -CI-algebra  $X$ . Then  $F$  is a  $\Gamma$ -filter of  $X$  if and only if for any  $a, b \in F, \gamma, \beta \in \Gamma$  and  $x \in X$ , if  $a\gamma(b\beta x) = 1$ , then  $x \in F$ .

**Proof.** Let  $F$  be a  $\Gamma$ -filter of  $\Gamma$ -CI-algebra  $X$ . Assume that  $a, b \in F, \gamma, \beta \in \Gamma$  and  $x \in X$  such that a

$$a\gamma(b\beta x) = 1 \in F.$$

By Definition 3.1, we have  $x \in F$ . For the converse assume that for any  $a, b \in F, \gamma, \beta \in \Gamma$  and  $x \in X$ ,  $a\gamma(b\beta x) = 1$  implies that  $x \in F$ . Suppose that a  $x \in F$  and  $x\gamma y \in F$ . By Proposition 2.5, we have  $a\gamma[(a\beta x)\alpha x] = 1$ . Then  $x \in F$  and hence  $F$  is a  $\Gamma$ -filter of  $X$ .

**Definition 3.4.** Let  $X$  be a  $\Gamma$ -CI-algebra and let  $a \in X, \gamma \in \Gamma$ . Define  $A(a\gamma)$  by

$$A(a\gamma) = \{1\} \cup \{x \in X : a\gamma x = 1\}$$

Then we call  $A(a\gamma)$  the initial section of the element  $a$ .

**Lemma 3.5.** Let  $X$  be a  $\Gamma$ -self distributive  $\Gamma$ -CI-algebra and  $x\gamma y = 1$  for all  $x, y \in X, \gamma \in \Gamma$ . If  $x \in A(a\gamma)$ , then  $y \in A(a\gamma)$ .

**Proof.** Let  $X$  be a  $\Gamma$ -self distributive  $\Gamma$ -CI-algebra  $X$  and  $x\gamma y = 1$  for all  $x, y \in X, \gamma \in \Gamma$ . Since  $x \in A(a\gamma)$ , we have  $a\gamma x = 1$ . Let  $a \in A$  and let  $x \in X$ . Then

$$\begin{aligned} a\gamma y &= a\gamma(1\beta y) \\ &= a\gamma((x\gamma y)\beta y) \\ &= (a\gamma(x\gamma y))\beta(a\gamma y) \\ &= ((a\gamma x)\gamma(a\gamma y))\beta(a\gamma y) \\ &= (1\gamma(a\gamma y))\beta(a\gamma y) \\ &= (a\gamma y)\beta(a\gamma y) \end{aligned}$$

$$= 1.$$

This implies  $y \in A(a\gamma)$ .

**Lemma 3.6.** Let  $X$  be a  $\Gamma$ -self distributive  $\Gamma$ -CI-algebra. Then  $A(a\gamma)$  is a  $\Gamma$ -filter of  $X$ .

**Proof.** Let  $X$  be a  $\Gamma$ -self distributive  $\Gamma$ -CI-algebra. By Definition 3.4, we have  $1 \in A(a\gamma)$ . Let  $x \in A(a\gamma)$  and  $x\beta y \in A(a\gamma)$ . Then  $a\gamma x = 1$  and  $a\gamma(x\beta y) = 1$  so that

$$\begin{aligned} a\gamma y &= 1\beta(a\gamma y) \\ &= (a\gamma x)\beta(a\gamma y) \\ &= a\gamma(x\beta y) \\ &= 1. \end{aligned}$$

This implies  $y \in A(a\gamma)$  and hence  $A(a\gamma)$  is a  $\Gamma$ -filter of  $X$ .

**Proposition 3.7.** Let  $X$  be a  $\Gamma$ -self-distributive  $\Gamma$ -CI-algebra and let  $x, y, y \in X, \gamma, \beta \in \Gamma$ . If  $z\gamma(x\beta y) = 1$  and  $z\gamma x = 1$ , then  $z\gamma y = 1$ .

**Proof.** Let  $X$  be a  $\Gamma$ -self-distributive  $\Gamma$ -CI-algebra and  $x, y, y \in X, \gamma, \beta \in \Gamma$ . Suppose that  $z\gamma(x\beta y) = 1$  and  $z\gamma x = 1$ . Then

$$\begin{aligned} z\gamma y &= 1\beta(z\gamma y) \\ &= (z\gamma x)\beta(z\gamma y) \\ &= z\gamma(x\beta y) \\ &= 1. \end{aligned}$$

Hence  $z\gamma y = 1$ .

**Theorem 3.8.** Let  $X$  be a  $\Gamma$ -CI-algebra,  $F$  a  $\Gamma$ -filter and  $x \in F, \gamma \in \Gamma$ . Then  $A(x\gamma) \subseteq F$ .

**Proof.** Let  $y \in A(x\gamma)$ . Then we have  $x\gamma y = 1$ . Since  $F$  is a  $\Gamma$ -filter of  $X$  and  $x \in X$ , we obtain  $y \in F$ . Therefore  $A(x\gamma) \subseteq F$ .

**Definition 3.9.** A  $\Gamma$ -CI-algebra  $X$  is said to be  $\Gamma$ -transitive if for all  $x, y, z \in X$  and  $\gamma, \alpha, \beta, \delta, \lambda \in \Gamma$ ,  $(y\gamma z)\alpha((x\beta y)\delta(x\lambda z)) = 1$ .

**Proposition 3.10.** If  $X$  is a  $\Gamma$ -self-distributive  $\Gamma$ -CI-algebra, then it is  $\Gamma$ -transitive.

**Proof.** For any  $x, y \in X$  and  $\gamma, \alpha, \beta, \delta, \lambda \in \Gamma$ , we have

$$\begin{aligned} (y\gamma z)\alpha((x\beta y)\delta(x\lambda z)) &= (y\gamma z)\alpha((x\beta y)\delta(x\beta z)) \\ &= (y\gamma z)\alpha(x\beta(y\delta z)) \\ &= x\alpha((y\gamma z)\beta(y\delta z)) \\ &= x\alpha((y\delta z)\beta(y\delta z)) \\ &= x\alpha 1 \\ &= 1. \end{aligned}$$

Hence  $X$  is a  $\Gamma$ -transitive.

**Definition 3.11.** Let  $X$  be a  $\Gamma$ -CI-algebra and let  $a, b \in X, \gamma, \beta \in \Gamma$ . Define  $A(a\gamma, b\beta)$  by

$$A(a\gamma, b\beta) = \{1\} \cup \{x \in X : a\gamma(b\beta x) = 1\}.$$

We call  $A(a\gamma, b\beta)$  an upper set of  $a$  and  $b$

**Lemma 3.12.** Let  $X$  be a  $\Gamma$ -CI-algebra and  $F$  be a  $\Gamma$ -filter of  $X$ . If  $a\gamma 1 = 1$ , then  $1 \in F_{a\gamma} = \{x : a\gamma x \in F\}$ , for any  $x \in X$  and  $\gamma \in \Gamma$ .

**Proof.** Suppose that  $F$  is a  $\Gamma$ -filter of  $X$ . Since  $1 = a\gamma 1$ , we have  $1 \in F_{a\gamma}$ .

**Theorem 4.13.** Let  $X$  be a  $\Gamma$ -self distributive  $\Gamma$ -CI-algebra and  $F$  be a  $\Gamma$ -filter of  $X$ . If  $a\gamma 1 = 1$ , then  $F_{a\gamma}$  is a  $\Gamma$ -filter, for any  $x \in X$  and  $\gamma \in \Gamma$ .

**Proof.** Suppose that  $F$  is a  $\Gamma$ -filter of  $X$ . By Lemma 3.12, we have  $1 \in F_{a\gamma}$ . Assume  $x \in F_{a\gamma}$  and  $x\beta y \in F_{a\gamma}$ . Then  $a\gamma x \in F$  and  $a\gamma(x\beta y) \in F$ . We have

$$a\gamma(x\beta y) = (a\gamma x)\beta(a\gamma y)$$

so that  $a\gamma y \in F$ . Therefore  $y \in F_{a\gamma}$  and hence  $F_{a\gamma}$  is a  $\Gamma$ -filter of  $X$ .

**Corollary 3.14.** Let  $X$  be a  $\Gamma$ -CI-algebra and  $F$  be a  $\Gamma$ -filter of  $X$ . If  $a\gamma 1 = 1$ , then  $A(a\gamma) \subseteq F_{a\gamma}$ , for any  $x \in X$  and  $\gamma \in \Gamma$ .

**Proof.** Suppose that  $F$  is a  $\Gamma$ -filter of  $X$ . Let  $x \in A(a\gamma)$ , for any  $x \in X$  and  $\gamma \in \Gamma$ . Then  $a\gamma x = 1 \in F$  so that  $x \in F_{a\gamma}$ . Hence  $A(a\gamma) \subseteq F_{a\gamma}$ .

#### 4. $\Gamma$ -LEFT-RIGHT DERIVATIONS IN $\Gamma$ -CI-ALGEBRAS

In this section, we introduce a relation " $\leq$ " on  $X$  by  $x \leq y$  if and only if  $x\gamma y = 1$  for all  $\gamma \in \Gamma$ .

**Remark** Let  $X$  be a  $\Gamma$ -CI-algebra  $X, a, b, x \in X$  and let  $\gamma, \beta \in \Gamma$ . Then

1.  $y \leq [(y\alpha x)\beta x]$
2.  $x \in A(a\gamma) \Leftrightarrow a \leq x$
3.  $x \in A(a\gamma, b\beta) \Leftrightarrow a \leq (b\beta x).$

**Lemma 4.1.** Let  $X$  be a  $\Gamma$ -CI-algebra and let  $x \in X$ . If  $1 \leq x$ , then  $x = 1$ .

**Proof.** Suppose that  $X$  is a  $\Gamma$ -CI-algebra. Since  $1 \leq x$ , we have  $1\gamma x = 1$ . By Definition 2.1, we have  $1\gamma x = x$ . Then  $x = 1$ .

**Lemma 4.2.** Let  $X$  be a  $\Gamma$ -CI-algebra. If  $X$  is a  $\Gamma$ -transitive, then for all  $x, y, z \in X, \gamma, \beta \in \Gamma$ , then  $y \leq z$  implies that  $x\beta y \leq x\gamma z$ .

**Proof.** Suppose that  $X$  is a  $\Gamma$ -transitive,  $x, y, z \in X$  and let  $\gamma, \beta, \alpha \in \Gamma$ . Since  $x \leq y$ , we have  $x\alpha y = 1$ . Then

$$(x\beta y)\lambda(x\gamma z) = (x\beta y)\lambda(x\gamma z)$$

$$\begin{aligned}
 &= 1\delta[(x\beta y)\lambda(x\gamma z)] \\
 &= (y\alpha z)\delta[(x\beta y)\lambda(x\gamma z)] \\
 &= 1.
 \end{aligned}$$

Hence  $x\beta y \leq x\gamma z$ .

**Proposition 4.3.** Let  $X$  be a  $\Gamma$ -self distributive  $\Gamma$ -CI-algebra. For all  $x, y, z \in X$  and  $\gamma, \beta, \alpha \in \Gamma$  if  $x\gamma 1 = 1$ , then

1. if  $x \leq y$ , then  $z\gamma x \leq z\gamma y$ ,
2.  $y\gamma z \leq (x\beta y)\gamma(x\alpha z)$ .

**Proof.** Let  $X$  be a  $\Gamma$ -self distributive  $\Gamma$ -CI-algebra,  $x, y, z \in X$  and  $\gamma, \beta, \alpha \in \Gamma$ .

1. Suppose that  $x \leq y$ . Then  $x\beta y = 1$ . We have

$$\begin{aligned}
 (z\gamma x)\alpha(z\gamma y) &= z\gamma(x\alpha y) \\
 &= z\gamma(x\beta y) \\
 &= z\gamma(x\gamma y) \\
 &= z\gamma 1 \\
 &= 1.
 \end{aligned}$$

Hence  $z\gamma x \leq z\gamma y$ .

2. Now consider

$$\begin{aligned}
 (y\gamma z)\delta[(x\beta y)\gamma(x\alpha z)] &= (y\gamma z)\delta[(x\beta y)\gamma(x\beta z)] \\
 &= (y\gamma z)\delta[x\beta(y\gamma z)] \\
 &= x\delta[(y\gamma z)\beta(y\gamma z)] \\
 &= x\delta 1 \\
 &= 1.
 \end{aligned}$$

Hence  $y\gamma z \leq (x\beta y)\gamma(x\alpha z)$ .

**Proposition 4.4.** Let  $X$  be a  $\Gamma$ -CI-algebra and  $F$  be a  $\Gamma$ -filter of  $X$ . Then for all  $x, y \in X$  and  $\gamma \in \Gamma$  the following statements hold:

1. If  $x \in F$  and  $x \leq y$ , then  $y \in F$ .
2. If  $X$  is a  $\Gamma$ -self distributive  $\Gamma$ -CI-algebra and  $x\beta y = 1$  for all  $x, y \in F$  and  $\gamma, \beta \in \Gamma$ , then  $x\gamma y \in F$ .

**Proof.** 1. Suppose that  $X$  is a  $\Gamma$ -CI-algebra and  $F$  is a  $\Gamma$ -filter of  $X$ . Let  $x \in F$  and  $x \leq y$ . Then  $x\gamma y = 1$  so that  $x\gamma y \in F$ . Therefore  $y \in F$ .

2. Suppose that  $X$  is a  $\Gamma$ -self distributive  $\Gamma$ -CI-algebra and  $x\beta y = 1$  for all  $x, y \in F$  and  $\gamma, \beta, \alpha \in \Gamma$ . We have

$$\begin{aligned}
 y\gamma(x\beta(x\alpha y)) &= x\gamma(y\beta(x\alpha y)) \\
 &= x\gamma((y\beta x)\alpha(y\beta y)) \\
 &= x\gamma((y\beta x)\alpha 1) \\
 &= [x\gamma(y\beta x)]\alpha(x\alpha 1) \\
 &= [(x\gamma y)\beta(x\beta x)]\alpha(x\alpha 1) \\
 &= [(x\gamma y)\beta 1]\alpha(x\alpha 1) \\
 &= [(x\beta y)\beta 1]\alpha(x\alpha 1)
 \end{aligned}$$

$$\begin{aligned}
 &= (1\beta 1)\alpha(x\alpha 1) \\
 &= 1\alpha(xy1) \\
 &= xy1 \\
 &= 1
 \end{aligned}$$

and hence  $y\gamma(x\beta(x\alpha y)) \in F$ . Since  $y \in F$ , we have  $x\beta(x\alpha y) \in F$ . It follows  $xyy = x\alpha y \in F$ . For elements  $x$  and  $y$  of a  $\Gamma$ -CI-algebra  $X$ , denote  $x \wedge y = (y\gamma x)\gamma x$ .

**Definition 4.5.** Let  $X$  be a  $\Gamma$ -CI-algebra. A mapping  $d: X \rightarrow X$  is a  $\Gamma$ -left-right derivation (briefly,  $\Gamma$ -(l, r)-derivation) of  $X$ , if it satisfies the identity  $d(xy\gamma y) = d(x)\gamma y \wedge xyd(y)$  for all  $x, y \in X$  and  $\gamma \in \Gamma$ . If  $d$  satisfies the identity  $d(xy\gamma y) = xyd(y) \wedge d(x)\gamma y$  for all  $x, y \in X$  and  $\gamma \in \Gamma$ , then  $d$  is a  $\Gamma$ -right-left derivation (briefly,  $\Gamma$ -(r, l)-derivation) of  $X$ . Moreover, if  $d$  is both a  $\Gamma$ -(l, r) and  $\Gamma$ -(r, l)-derivation, then  $d$  is a  $\Gamma$ -derivation of  $X$ .

**Proposition 4.6.** If  $d$  is a  $\Gamma$ -(l,r)-derivation of  $\Gamma$ -self-distributive  $X$ , then,  $d(1) = 1$ .

**Proof.** Let  $X$  be a  $\Gamma$ -CI-algebra and let  $d$  be a  $\Gamma$ -left-right derivation of  $X$ . Then

$$\begin{aligned}
 d(1) &= d(1\gamma 1) \\
 &= d(1)\gamma 1 \wedge 1\gamma d(1) \\
 &= d(1)\gamma 1 \wedge d(1) \\
 &= [d(1)\beta(d(1)\gamma 1)]\beta(d(1)\gamma 1) \\
 &= [(d(1)\beta d(1))\gamma(d(1)\beta 1)]\beta(d(1)\gamma 1) \\
 &= [1\gamma(d(1)\beta 1)]\beta(d(1)\gamma 1) \\
 &= (d(1)\beta 1)\beta(d(1)\gamma 1) \\
 &= (d(1)\gamma 1)\beta(d(1)\gamma 1) \\
 &= 1.
 \end{aligned}$$

Hence  $d(1) = 1$ .

**Definition 4.7.** A  $\Gamma$ -derivation  $d$  of a  $\Gamma$ -CI-algebra  $X$  is said to be  $\Gamma$ -regular if  $d(1) = 1$ .

**Corollary 4.8.** A  $\Gamma$ -(l,r)-derivation  $d$  of a  $\Gamma$ -self-distributive  $X$  is  $\Gamma$ -regular.

**Proof.** Straightforward.  $\square$

**Proposition 4.9.** Let  $d$  be a  $\Gamma$ -(l,r)-derivation of a  $\Gamma$ -self-distributive  $X$ . Then  $d(x) = x \wedge d(x)$  for all  $x \in X$ .

**Proof.** Let  $d$  be a  $\Gamma$ -(l,r)-derivation of a  $\Gamma$ -self-distributive  $X$  and  $x \in X, \gamma \in \Gamma$ . Then

$$\begin{aligned}
 d(x) &= d(1\gamma x) \\
 &= d(1)\gamma x \wedge 1\gamma d(x) \\
 &= d(1)\gamma x \wedge d(x) \\
 &= 1\gamma x \wedge d(x)
 \end{aligned}$$

$$= x \wedge d(x).$$

Hence  $d(x) = x \wedge d(x)$ .

**Definition 4.10.** Let  $X$  a  $\Gamma$ -CI-algebra. A map  $f: X \rightarrow X$  is called endomorphism if  $f(x\gamma y) = f(x)\gamma f(y)$  for all  $x, y \in X$  and  $\gamma \in \Gamma$ .

**Definition 4.11.** Let  $f$  be an endomorphism of a  $\Gamma$ -CI-algebra  $X$ . A map  $d_f: X \rightarrow X$  satisfying the identity

$$d_f(x\gamma y) = d_f(x)\gamma f(y) \wedge f(x)\gamma d_f(y)$$

is called a  $\Gamma$ -left-right  $f$ -derivation (briefly,  $\Gamma$ -(l; r)- $f$ -derivation) of  $X$ .

**Proposition 4.12.** Let  $d_f$  be a  $\Gamma$ -(1,r)-derivation of a  $\Gamma$ -self-distributive  $X$ . Then  $d_f(x) = 1$  for all  $x \in X$ .

**Proof.** Let  $d_f$  be a  $\Gamma$ -(1,r)-derivation of a  $\Gamma$ -self-distributive  $X$ . For all  $\gamma, \beta \in \Gamma$  so that

$$\begin{aligned} d_f(1) &= d_f(1\gamma 1) \\ &= d_f(1)\gamma f(1) \wedge f(1)\gamma d_f(1) \\ &= d_f(1)\gamma f(1) \wedge f(1)\gamma d_f(1) \\ &= d_f(1)\gamma f(1\gamma 1) \wedge f(1\gamma 1)\gamma d_f(1) \\ &= d_f(1)\gamma [f(1)\gamma f(1)] \wedge [f(1)\gamma f(1)]\gamma d_f(1) \\ &= d_f(1)\gamma 1 \wedge 1\gamma d_f(1) \\ &= d_f(1)\gamma 1 \wedge d_f(1) \\ &= [d_f(1)\beta(d_f(1)\gamma 1)]\beta(d_f(1)\gamma 1) \\ &= [(d_f(1)\beta d_f(1))\gamma(d_f(1)\beta 1)]\beta(d_f(1)\gamma 1) \\ &= [1\gamma(d_f(1)\beta 1)]\beta(d_f(1)\gamma 1) \\ &= (d_f(1)\beta 1)\beta(d_f(1)\gamma 1) \\ &= (d_f(1)\gamma 1)\beta(d_f(1)\gamma 1) \\ &= 1. \end{aligned}$$

Hence  $d_f(1) = 1$

**Proposition 4.13.** Let  $d_f$  be a  $\Gamma$ -(1,r)-derivation of a  $\Gamma$ -self-distributive  $X$ . Then  $d_f(x) = d_f(x) \wedge f(x)$  for all  $x \in X$ .

**Proof.** Let  $d_f$  be a  $\Gamma$ -(1,r)-derivation of a  $\Gamma$ -self-distributive  $X$ . For all  $\gamma, \beta \in \Gamma$  so that

$$\begin{aligned} d_f(x) &= d_f(1\gamma x) \\ &= d_f(1)\gamma f(x) \wedge f(1)\gamma d_f(x) \\ &= 1\gamma f(x) \wedge 1\gamma d_f(x) \\ &= f(x) \wedge d_f(x). \end{aligned}$$

Hence  $d_f(x) = d_f(x) \wedge f(x)$ .

**Theorem 4.14.** If  $d_f$  is a  $\Gamma$ -regular  $\Gamma$ -(l; r)-f-derivation of a  $\Gamma$ -CI-algebra  $X$ , then  $d_f(x) = f(x) \wedge d_f(x)$

**Proof.** Let  $d_f$  be a  $\Gamma$ -regular  $\Gamma$ -(l; r)-f-derivation of a  $\Gamma$ -CI-algebra  $X$ . For all  $\gamma, \beta \in \Gamma$  so that

$$\begin{aligned} d_f(x) &= d_f(1\gamma x) \\ &= d_f(1)\gamma f(x) \wedge f(1)\gamma d_f(x) \\ &= 1\gamma f(x) \wedge 1\gamma d_f(x) \\ &= f(x) \wedge d_f(x). \end{aligned}$$

Hence  $d_f(x) = f(x) \wedge d_f(x)$ .

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