

On Quasi-Left Primary and Quasi-Primary Γ -ideals in Γ -AG-Groupoids

Pairote Yiarayong

Department of Mathematics,
Faculty of Science and Technology,
Pibulsongkram Rajabhat University,
Phitsanuloke 65000, Thailand
Email: pairote0027 [AT] hotmail.com

ABSTRACT— *The purpose of this paper is to introduce the notion of a quasi-primary ideals in Γ -AG-groupoids, we study quasi-primary and quasi-left primary ideals in Γ -AG-groupoids. Some characterizations of quasi-primary and quasi-left primary ideals are obtained. Moreover, we investigate the relationships between quasi-primary and quasi-left primary ideals in Γ -AG-groupoids. Finally, we obtain the necessary and sufficient conditions of a quasi-primary ideal to be a quasi-left primary ideal in Γ -AG-groupoids.*

Keywords— Γ -AG-groupoid, Γ -LA-semigroup, Γ -ideal, quasi-primary ideal, quasi-left primary ideal.

1. INTRODUCTION

Abel-Grassmann's groupoid (AG-groupoid) is a generalization of semigroup theory with wide range of usages in theory of flocks [6]. The fundamentals of this non-associative algebraic structure were first discovered by Kazim and Naseeruddin (1972). A groupoid S is called an AG-groupoid if it satisfies the left invertive law:

$$(ab)c = (cb)a$$

for all $a, b, c \in S$. It is interesting to note that an AG-groupoid with right identity becomes a commutative monoid [5]. This structure is closely related to a commutative semigroup. Because of containing a right identity, an AG-groupoid becomes a commutative monoid [5]. A left identity in an AG-groupoid is unique [5]. It lies between a groupoid and a commutative semigroup with wide range of applications in theory of flocks [6]. Ideals in AG-groupoids have been discussed in [4]. In 1981, the notion of Γ -semigroups was introduced by M. K. Sen. A groupoid S is called a Γ -AG-groupoid if it satisfies the left invertive law:

$$(a\gamma b)\delta c = (c\gamma b)\delta a$$

for all $a, b, c \in S$ and $\gamma, \delta \in \Gamma$ [3]. This structure is also known as a left almost semigroup (LA-semigroup). In this paper, we are going to investigate some interesting properties of recently discovered classes, namely Γ -AG-groupoid S always satisfies the Γ -medial law:

$$(a\gamma b)\beta(c\delta d) = (a\gamma c)\beta(b\delta d)$$

for all $a, b, c, d \in S$ and $\gamma, \beta, \delta \in \Gamma$ [3], while a Γ -AG-groupoid S with left identity e always satisfies Γ -paramedial law:

$$(a\gamma b)\beta(c\delta d) = (d\gamma c)\beta(b\delta a)$$

for all $a, b, c, d \in S$ and $\gamma, \beta, \delta \in \Gamma$ [3]. Recently T. Shah and I. Rehman have discussed Γ -Ideals and Γ -Bi-Ideals in Γ -AG-Groupoids.

In this paper we characterize the Γ -AG-groupoid. We investigate the relationships between quasi-primary and quasi-left primary ideals in Γ -AG-groupoids.

2. BASIC PROPERTIES

In this section we refer to [10, 11, 12, 13] for some elementary aspects and quote few definitions, and essential examples to step up this study. For more details we refer to the papers in the references.

Example 2.1. [10, 11] (1). Let S be an arbitrary AG-groupoid and Γ any non-empty set. Define a mapping $S \times \Gamma \times S \rightarrow S$; by $a\gamma b = ab$ for all $a, b \in S$ and $\gamma \in \Gamma$. It is easy to see that S is a Γ -AG-groupoid.

(2). Let $\Gamma = \{1, 2, 3\}$. Define a mapping $\square \times \Gamma \times \square \rightarrow \square$ by $a\gamma b = b - \gamma - a$ for all $a, b \in \square$ and $\gamma \in \Gamma$ where " $-$ " is a usual subtraction of integers. Then $\square$ is a Γ -AG-groupoid.

Lemma 2.2. [10, 11] Every Γ -AG-groupoid is Γ -medial.

Lemma 2.3. [10, 11] Let S be a Γ -AG-groupoid with a left identity, then $a\gamma(b\alpha c) = b\gamma(a\alpha c)$ for all $a, b, c \in S$ and $\gamma, \alpha \in \Gamma$.

Definition 2.4. [10, 11] Let S be a Γ -AG-groupoid. A nonempty subset A of S is called a sub Γ -AG-groupoid of S if $A\Gamma A \subseteq A$.

Definition 2.5. [10, 11] A sub Γ -AG-groupoid A of S is called a left (right) Γ -ideal of S if $S\Gamma A \subseteq A$ ($A\Gamma S \subseteq A$) and is called an Γ -ideal if it is left as well as right Γ -ideal.

Lemma 2.6. [10, 11] If a Γ -AG-groupoid S has a left identity, then every right Γ -ideal is a left Γ -ideal.

Lemma 2.7. [10, 11] If A is a left Γ -ideal of a Γ -AG-groupoid S with left identity, and if for any $a \in S$, there exists $\gamma \in \Gamma$, then $a\gamma A$ is a left Γ -ideal of S .

Lemma 2.8. [10, 11] If A is a proper right (left) Γ -ideal of a Γ -AG-groupoid S with left identity e , then $e \notin A$.

Lemma 2.9. [13] If S is a Γ -AG-groupoid with left identity e , then $a\gamma b = a\beta b$ for all $a, b \in S$ and $\gamma, \beta \in \Gamma$.

Lemma 2.10. Let S be a Γ -AG-groupoid with left identity, and let B be a left Γ -ideal of S . Then $A\Gamma B = \{a\gamma b : a \in A, b \in B, \gamma \in \Gamma\}$ is a left Γ -ideal in S , where $\emptyset \neq A \subseteq S$.

Lemma 2.11. Let S be a Γ -AG-groupoid with left identity and let $a \in S$. Then

$$a^2\gamma S = \{a^2\gamma s = (a\beta a)\gamma s : s \in S\}$$

is a Γ -ideal in S , where $\gamma, \beta \in \Gamma$.

Lemma 2.12. Let S be a Γ -AG-groupoid with left identity, and let A be a left Γ -ideal of S . Then $(A : \gamma : r)$ is a left Γ -ideal in S , where $(A : \gamma : r) = \{a \in S : r\gamma a \in A\}$.

Remark. Let S be a Γ -AG-groupoid and let A be a left Γ -ideal of S . It is easy to verify that $A \subseteq (A : \gamma : r)$.

Lemma 2.13. Let S be a Γ -AG-groupoid with left identity, and let A, B be left Γ -ideals of S . Then $(A : \Gamma : B)$ is a left Γ -ideal in S , where $(A : \Gamma : B) = \{r \in S : B\Gamma r \subseteq A\}$.

Remark. Let S be a Γ -AG-groupoid and let A, B, C be left Γ -ideals of S . It is easy to verify that $(A : \Gamma : C) \subseteq (A : \Gamma : B)$, where $B \subseteq C$.

3. QUASI-LEFT PRIMARY AND LEFT PRIMARY Γ -IDEALS

We start with the following theorem that gives a relation between Γ -primary and quasi Γ -primary ideal in Γ -AG-groupoid. Our starting points are the following definitions:

Definition 3.1. A Γ -ideal P is called left quasi-primary if $A\Gamma B \subseteq P$ implies that

$$(((A\Gamma A)\Gamma A)\Gamma \dots)\Gamma A = A^n \subseteq P \text{ or } (((B\Gamma B)\Gamma B)\Gamma \dots)\Gamma B = B^n \subseteq P$$

for some positive integer n , where A and B are two Γ -ideals of S .

Definition 3.2. A left Γ -ideal P is called quasi-left primary if $A\Gamma B \subseteq P$ implies that

$$(((A\Gamma A)\Gamma A)\Gamma \dots)\Gamma A = A^n \subseteq P \text{ or } (((B\Gamma B)\Gamma B)\Gamma \dots)\Gamma B = B^n \subseteq P$$

for some positive integer n , where A and B are two left Γ -ideals of S .

Remark. It is easy to see that every quasi-left primary ideal is quasi-primary.

Lemma 3.3. If S is a Γ -AG-groupoid with left identity, then a left Γ -ideal P of S is quasi-left primary if and only if $a\gamma(S\beta b) \subseteq P$ implies that

$$(((a\delta a)\delta a)\delta \dots)\delta a = a^n \in P \text{ or } (((b\delta b)\delta b)\delta \dots)\delta b = b^n \in P$$

for some positive integer n , where $\gamma, \beta, \delta \in \Gamma$ and $a, b \in S$.

Proof. Let P be a quasi-left primary left ideal of a Γ -AG-groupoid S with left identity. Now suppose that $a\gamma(S\beta b) \subseteq P$. Then by Definition of left Γ -ideal, we get $ST(a\gamma(S\beta b)) \subseteq STP \subseteq P$ that is,

$$\begin{aligned} ST(a\gamma(S\beta b)) &= (S\delta S)\Gamma(a\gamma(S\beta b)) \\ &= (S\delta a)\Gamma(S\gamma(S\beta b)) \\ &= (S\delta a)\Gamma((STS)\gamma(S\beta b)) \\ &= (S\delta a)\Gamma((b\beta S)\gamma(STS)) \\ &= (S\delta a)\Gamma((b\beta S)\gamma(S)) \\ &= (S\delta a)\Gamma((S\beta S)\gamma b) \\ &= (S\delta a)\Gamma(S\gamma b) \end{aligned}$$

for all $\delta \in \Gamma$. Since $ST(a\gamma(S\beta b)) \subseteq P$ and $ST(a\gamma(S\beta b)) = (S\delta a)\Gamma(S\gamma b)$, we have $(S\delta a)\Gamma(S\gamma b) \subseteq P$ so that $a^n = (e\delta a)^n \in (S\delta a)^n \subseteq P$ or $b^n = (eb)^n \in (S\gamma b)^n \subseteq P$, for some positive integer n . Conversely, assume that if $a\gamma(S\beta b) \subseteq P$ implies that $a^n \in P$ or $b^n \in P$ for some positive integer n , where $\gamma, \beta \in \Gamma$ and $a, b \in S$. Suppose that $A\Gamma B \subseteq P$, where A and B are left Γ -ideals of S such that $A \not\subseteq P$. Then there exists $x \in A$ such that $x^n \notin P$, for all positive integer n . Now

$$x\gamma(S\beta y) \subseteq A\Gamma(STB) \subseteq A\Gamma B \subseteq P,$$

for all $y \in B$. So by hypothesis, $y^n \in P$ for all $y \in B$ implies that $B^n \subseteq P$. Hence P is quasi-left primary ideal in S .

Lemma 3.4. If S is a Γ -AG-groupoid with left identity, then a left Γ -ideal P of S is quasi-left primary if and only if $(S\gamma a)\delta(S\beta b) \subseteq P$ implies that $a^n \in P$ or $b^n \in P$ for some positive integer n , where $\gamma, \beta, \delta \in \Gamma$ and $a, b \in S$.

Proof. Let P be a quasi-left primary ideal of a Γ -AG-groupoid S with left identity. Now suppose that $(S\gamma a)\delta(S\beta b) \subseteq P$. Then by Definition of left ideal, we get

$$(S\gamma a)\delta(S\beta b) = (S\gamma S)\delta(a\beta b)$$

$$\begin{aligned} &= S\delta(a\beta b) \\ &= a\delta(S\beta b) \end{aligned}$$

that is $a\delta(S\beta b) = (S\gamma a)\delta(S\beta b) \subseteq P$. By Lemma 3.3, we have $a^n \in P$ or $b^n \in P$ for some positive integer n .

Conversely, assume that if $(S\gamma a)\delta(S\beta b) \subseteq P$, then $a^n \in P$ or $b^n \in P$ for some positive integer n , where $\gamma, \beta, \delta \in \Gamma$ and $a, b \in S$. Let $a\delta(S\beta b) \subseteq P$. Now consider

$$a\delta(S\beta b) = (S\gamma a)\delta(S\beta b) \subseteq P.$$

By using given assumption, if $a\delta(S\beta b) \subseteq P$, then $a^n \in P$ or $b^n \in P$ for some positive integer n . Then by Lemma 3.3, we have P is a quasi-left primary ideal in S .

Theorem 3.5. If S is a Γ -AG-groupoid with left identity, then a left Γ -ideal P of S is quasi-left primary if and only if $a\gamma b \in P$ implies that $a^n \in P$ or $b^n \in P$ for some positive integer n , where $\gamma \in \Gamma$ and $a, b \in S$.

Proof. Let P be a left Γ -ideal of a Γ -AG-groupoid S with left identity. Now suppose that $a\gamma b \in P$. Then by Definition of left ideal, we get

$$\begin{aligned} (S\alpha a)\beta(S\gamma b) &= (S\alpha S)\beta(a\gamma b) \\ &= S\beta(a\gamma b) \\ &\subseteq STP \\ &\subseteq P. \end{aligned}$$

By Lemma 3.4, we have $a^n \in P$ or $b^n \in P$ for some positive integer n . Conversely, the proof is easy.

Theorem 3.6. Let S be a Γ -AG-groupoid, and let A be a quasi-left primary ideal of S . Then $(A : \gamma : r)$ is a quasi-left primary ideal in S , where $\gamma \in \Gamma$ and $r \in S$.

Proof. Assume that A is a quasi-left primary ideal of S . By Lemma 2.12, we have $(A : \gamma : r)$ is a left ideal in S . Let $a\beta b \in (A : \gamma : r)$. Suppose that $b^n \notin (A : \gamma : r)$, for all positive integer n . Since $a\beta b \in (A : \gamma : r)$, we have $r\gamma(a\beta b) \in A$ so that $a\gamma(r\beta b) \in A$. By Theorem 3.5, we have $a^n \in A \subseteq (A : \gamma : r)$ or $(r\beta b)^n \in A$, for some positive integer n . Therefore $a^n \in (A : \gamma : r)$ and hence $(A : \gamma : r)$ is a quasi-left primary ideal in S .

Theorem 3.7. Let S be a Γ -AG-groupoid with left identity e and let P be a quasi-primary ideal of S . If $(S\gamma a^2)\alpha(S\beta b^2) \subseteq P$, then $a^n \in P$ or $b^n \in P$, for some positive integer n , where $\gamma \in \Gamma$ and $a, b \in S$.

Proof. Let P be a quasi-primary ideal of a Γ -AG-groupoid S with left identity. Suppose that $b^n \notin P$, for all positive integer n . Now assume that $(S\gamma a^2)\alpha(S\beta b^2) \subseteq P$. Then by Definition of left Γ -ideal, we get

$$\begin{aligned}(S\gamma a^2)\alpha(S\beta b^2) &= ((S\beta b^2)\gamma a^2)\alpha S \\ &= ((a^2\beta b^2)\gamma S)\alpha S \\ &= (S\gamma S)\alpha(a^2\beta b^2) \\ &= a^2\alpha((S\gamma S)\beta b^2) \\ &= a^2\alpha((b^2\gamma S)\beta S) \\ &= (b^2\gamma S)\alpha(a^2\beta S)\end{aligned}$$

that is $(b^2\gamma S)\alpha(a^2\beta S) \subseteq P$. By Lemma 2.11, we have $a^2\beta S$ and $b^2\gamma S$ are Γ -ideals in S so that

$$\begin{aligned}a^2 &= a\lambda a \\ &= (e\chi a)\lambda a \\ &= (a\chi a)\lambda e \\ &= (a\lambda a)\beta e \\ &= a^2\beta e \in a^2\beta S \subseteq P\end{aligned}$$

or

$$\begin{aligned}b^2 &= b\lambda b \\ &= (e\chi b)\lambda b \\ &= (b\chi b)\lambda e \\ &= (b\lambda b)\gamma e \\ &= b^2\gamma e \in b^2\gamma S \subseteq P\end{aligned}$$

for all $\chi \in \Gamma$. Therefore $a^n \in P$, for some positive integer n .

Corollary 3.8. Let S be a Γ -AG-groupoid with left identity, and let P be a quasi-primary ideal of S . If $b^2\gamma a^2 \in P$, then $a^n \in P$ or $b^n \in P$, for some positive integer n .

Proof. Let P be a quasi-primary ideal of an AG-groupoid S with left identity. Suppose that $b^n \notin P$, for all positive integer n . Now assume that $b^2\gamma a^2 \in P$. Then by Definition of left Γ -ideal, we get

$$\begin{aligned}
 (a^2\beta S)\alpha(b^2\gamma S) &= b^2\alpha((a^2\beta S)\gamma S) \\
 &= b^2\alpha((S\beta S)\gamma a^2) \\
 &= (S\beta S)\alpha(b^2\gamma a^2) \\
 &= S\alpha(b^2\gamma a^2) \\
 &\subseteq STP \\
 &\subseteq P
 \end{aligned}$$

that is $(a^2\beta S)\alpha(b^2\gamma S) \subseteq P$. It is easy to see that $a^n \in P$, for some positive integer n .

Definition 3.9. A Γ -AG-groupoid S is called Γ -AG-3-band if its every element satisfies

$$a\alpha(a\beta a) = (a\alpha a)\beta a = a.$$

Proposition 3.10. [13] Every left identity in a Γ -AG-3-band is a right identity.

Lemma 3.11. [13] If a Γ -AG-3-band S has a left identity, then every left Γ -ideal is a Γ -ideal.

Theorem 3.12. Let S be a Γ -AG-3-band with left identity. Then P is a quasi-left primary ideal in S if and only if S is a quasi-primary ideal in S .

Proof. The proof is straightforward. □

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