

SOME BASIC PROPERTIES OF Γ -PRIMARY AND QUASI Γ -PRIMARY IDEALS IN Γ -AG-GROUPOIDS

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Abstract

The purpose of this paper is to introduce the notion of a Γ -primary ideals in Γ -AG-groupoids, and Γ -ideals, Γ -primary ideals, and quasi Γ -primary ideals in Γ -AG-groupoids are studied. Some characterizations of Γ -primary and quasi Γ -primary ideals are obtained. Moreover, the relationships between Γ -primary and quasi Γ -primary ideals in Γ -AG-groupoids are investigated. Finally, the necessary and sufficient conditions of a Γ -primary ideal to be a quasi Γ -primary ideal in Γ -AG-groupoids are obtained.

Keywords: Γ -AG-groupoid, Γ -LA-semigroup, Γ -ideal, quasi Γ -primary ideal, Γ -primary ideal

Introduction

Abel-Grassmann's groupoid (AG-groupoid) is a generalization of semigroup theory with a wide range of usages in the theory of flocks (Naseeruddin, 1970). The fundamentals of this non-associative algebraic structure were first discovered by Kazim and Naseeruddin (1972). A groupoid S is called an AG-groupoid if it satisfies the left invertive law:

$$(ab)c = (cb)a$$

for $a, b, c \in S$ all. This structure is closely related with a commutative semigroup because if an AG-groupoid contains a right identity, then it becomes a commutative monoid (Mushtaq and Yousuf, 1978). A left identity in an AG-groupoid is unique (Mushtaq and

Yousuf, 1978). It lies between a groupoid and a commutative semigroup with a wide range of applications in the theory of flocks (Naseeruddin, 1970). Ideals in AG-groupoids have been discussed in Mushtaq and Yousuf (1988). The notion of Γ -semigroups was introduced by Sen (1981). A groupoid S is called a Γ -AG-groupoid if it satisfies the left invertive law:

$$(a\gamma b)\delta c = (c\gamma b)\delta a$$

for all $a, b, c, d \in S$ and $\gamma, \delta \in \Gamma$ (Khan *et al.*, 2013). This structure is also known as a left almost semigroup (LA-semigroup). In this paper there are investigated some interesting properties of newly discovered classes of

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of Γ -AG-groupoid S always satisfies the Γ -medial law:

$$(\alpha\gamma b)\beta(c\delta d) = (\alpha\gamma c)\beta(b\delta d)$$

for all $a, b, c, d \in S$ and $\gamma, \beta, \delta \in \Gamma$ (Khan *et al.*, 2013), while a Γ -AG-groupoid S with a left identity e always satisfies the Γ -paramedial law:

$$(\alpha\gamma b)\beta(c\delta d) = (d\gamma c)\beta(b\delta a)$$

for all $a, b, c, d \in S$ and $\gamma, \beta, \delta \in \Gamma$ (Khan *et al.*, 2013). Recently Shah and Rehman, (2010b) discussed Γ -Ideals and Γ -bi-ideals in Γ -AG-groupoids.

In this paper, the Γ -AG-groupoids are characterized and the relationships between Γ -primary and quasi Γ -primary ideals in Γ -AG-groupoids are investigated.

Basic Results

In this section, reference is made to Shah and Rehman, (2010a) for some elementary aspects and to quote a few definitions and examples which are essential to step up this study. For more details, reference should be made to the literature in the References.

Example 1. (Shah and Rehman, 2010a) (1). Let S be an arbitrary AG-groupoid and Γ any non-empty set. Define a mapping $S \times \Gamma \times S \rightarrow S$; by $\alpha\gamma b = ab$ for all $a, b \in S$ and $\gamma \in \Gamma$. It is easy to see that S is a Γ -AG-groupoid.

(2). Let $\Gamma = \{1, 2, 3\}$. Define a mapping $\mathbb{Z} \times \Gamma \times \mathbb{Z} \rightarrow \mathbb{Z}$ by $\alpha\gamma b = b - \gamma - a$ for all $a, b \in \mathbb{Z}$ and $\gamma \in \Gamma$ where "-" is a usual subtraction of integers. Then $\mathbb{Z}$ is a Γ -AG-groupoid.

Lemma 2. (Shah and Rehman, 2010a) Every Γ -AG-groupoid is Γ -medial.

Lemma 3. (Shah and Rehman, 2010a) Let S be a Γ -AG-groupoid with a left identity, then $\alpha\gamma(bac) = b\gamma(aac)$ for all $a, b, c \in S$ and $\gamma, \alpha \in \Gamma$.

Definition 4. (Shah and Rehman, 2010a) Let S be a Γ -AG-groupoid. A non-empty subset A of S is called a sub Γ -AG-groupoid of S if $A\Gamma A \subseteq A$.

Definition 5. (Shah and Rehman, 2010a) A sub Γ -AG-groupoid A of S is called a left (right) Γ -ideal of S if $S\Gamma A \subseteq A$ and is called a Γ -ideal if it is a left as well as a right Γ -ideal.

Lemma 6. (Shah and Rehman, 2013) If a Γ -AG-groupoid S has a left identity, then every right Γ -ideal is a left Γ -ideal.

Lemma 7. (Shah and Rehman, 2013) If A is a left Γ -ideal of a Γ -AG-groupoid S with a left identity, and if, for any $a \in S$, there exists $\gamma \in \Gamma$, then $a\gamma A$ is a left Γ -ideal of S .

Lemma 8. (Shah and Rehman, 2013) If A is a proper right (left) Γ -ideal of a Γ -AG-groupoid S with a left identity e , then $e \notin A$.

Γ -Ideals in Γ -AG-groupoids

The results of the following lemmas seem to play an important role in the study of Γ -AG-groupoids; these facts will be used so frequently that normally no reference to this lemma will be made.

Lemma 9. If S is a Γ -AG-groupoid with a left identity, then $\alpha\gamma b = a\beta b$ for all $a, b \in S$ and $\gamma, \beta \in \Gamma$.

Proof. Let S be a Γ -AG-groupoid and e be the left identity of S , $a, b \in S$ and let $\gamma, \beta \in \Gamma$; therefore we have

$$\begin{aligned} \alpha\gamma b &= \alpha\gamma(e\beta b) \\ &= e\gamma(e\beta b) \\ &= a\beta b. \end{aligned}$$

Hence, $\alpha\gamma b = a\beta b$.

Lemma 10. Let S be a Γ -AG-groupoid with a left identity, and let B be a left Γ -ideal of S . Then $A\Gamma B = \{\alpha\gamma b : a \in A, b \in B, \gamma \in \Gamma\}$ is a left Γ -ideal in S , where $\emptyset \neq A \subseteq S$.

Proof. Suppose that S is a Γ -AG-groupoid with a left identity. Let B be a left Γ -ideal of S . Then,

$$S\Gamma (A\Gamma B) = A\Gamma (S\Gamma B) \subseteq A\Gamma B.$$

By Definition of a left Γ -ideal, we get $A\Gamma B$

is a left Γ -ideal in S .

Lemma 11. Let S be a Γ -AG-groupoid with a left identity and let $a \in S$. Then $a^2\gamma S$ is a Γ -ideal in S , where $\gamma \in \Gamma$.

Proof. By Lemma 7, we have $a^2\gamma S$ is a left Γ -ideal of S . Now consider

$$\begin{aligned}(a^2\gamma r)as &= (s\gamma r)\alpha a^2 \\ &= a^2\alpha(r\gamma s) \\ &= a^2\gamma(r\gamma s) \in a^2\gamma S\end{aligned}$$

for all $r, s \in S$ and $\alpha \in \Gamma$. Therefore, $a^2\gamma S$ is a Γ -ideal in S .

Lemma 12. Let S be a Γ -AG-groupoid with a left identity, and let A be a left Γ -ideal of S . Then $(A : \gamma : r)$ is a left Γ -ideal in S , where $(A : \gamma : r) = \{a \in S : r\gamma a \in A\}$.

Proof. Suppose that S is a Γ -AG-groupoid. Let $s \in S$, $a \in (A : \gamma : r)$. Then $r\gamma a \in A$ and so that

$$r\gamma(s\alpha a) = r\alpha(s\gamma a) = s\alpha(r\gamma a) \in s\alpha A \subseteq A$$

for all $\alpha \in \Gamma$. Therefore, $s\alpha a \in (A : \gamma : r)$ so that $S\Gamma(A : \gamma : r) \subseteq (A : \gamma : r)$. Hence $(A : \gamma : r)$ is a left Γ -ideal in S .

Remark Let S be a Γ -AG-groupoid and let A be a left Γ -ideal of S . It is easy to verify that $A \subseteq (A : \gamma : r)$.

Lemma 13. Let S be a Γ -AG-groupoid with left identity, and let A, B be left ideals of S . Then $(A : \Gamma : B)$ is a Γ -left ideal in S , where $(A : \Gamma : B) = \{r \in S : B\Gamma r \subseteq A\}$.

Proof. Suppose that S is a Γ -AG-groupoid. Let $s \in S$ and let $a \in (A : \Gamma : B)$. Then $B\Gamma a \subseteq A$ so that

$$B\alpha(s\gamma a) = s\alpha(B\gamma a) \subseteq s\alpha(B\Gamma a) \subseteq s\alpha A \subseteq A$$

for all $\gamma, \beta \in \Gamma$. Therefore, $s\gamma a \in (A : \Gamma : B)$ so that $S\Gamma(A : \Gamma : B) \subseteq (A : \Gamma : B)$. Hence, $(A : \Gamma : B)$ is a left Γ -ideal in S .

Remark Let S be a Γ -AG-groupoid and let A, B, C be left Γ -ideals of S . It is easy to

verify that $(A : \Gamma : C) \subseteq (A : \Gamma : B)$, where $B \subseteq C$.

Γ -Primary Ideals in Γ -AG-groupoids

When starting with the following theorem that gives a relation between Γ -primary and quasi Γ -primary ideals in a Γ -AG-groupoid, the starting point is the following definition:

Definition 14. A Γ -ideal P is called Γ -primary if $A\Gamma B \subseteq P$ implies that $A \subseteq P$ or $B^n \subseteq P$ for some positive integer n , where A, B is a Γ -ideals of S .

Definition 15. A left Γ -ideal P is called a quasi Γ -primary if $A\Gamma B \subseteq P$ implies that $A \subseteq P$ or $B^n \subseteq P$ for some positive integer n , where A, B is a left Γ -ideals of S .

Remark It is easy to see that every quasi Γ -primary ideal is Γ -primary.

Lemma 16. If S is a Γ -AG-groupoid with a left identity, then a left Γ -ideal P of S is a quasi Γ -primary if, and only if, $a\gamma(S\alpha b) \subseteq P$ implies that $a \in P$ or $b^n \in P$ for some positive integer n , where $\gamma, \beta \in \Gamma$ and $a, b \in S$.

Proof. Let P be a quasi Γ -primary left ideal of a Γ -AG-groupoid S with a left identity. Now suppose that $a\gamma(S\beta b) \subseteq P$. Then, by definition of a left Γ -ideal, we get $S\Gamma(a\gamma(S\beta b)) \subseteq S\Gamma P \subseteq P$ that is

$$\begin{aligned}S\Gamma(a\gamma(S\beta b)) &= (S\delta S)\Gamma(a\gamma(S\beta b)) \\ &= (S\delta a)\Gamma(S\gamma(S\beta b)) \\ &= (S\delta a)\Gamma((S\Gamma S)\gamma(S\beta b)) \\ &= (S\delta a)\Gamma((b\beta S)\gamma(S\Gamma S)) \\ &= (S\delta a)\Gamma((b\beta S)\gamma(S)) \\ &= (S\delta a)\Gamma(S\beta S)\gamma b \\ &= (S\delta a)\Gamma(S\gamma b)\end{aligned}$$

for all $\delta \in \Gamma$. Then, $(S\delta a)\Gamma(S\delta b) = a\gamma(S\beta b) \subseteq P$ so that $a \in \langle a \rangle \subseteq P$ or $a^n \in \langle a \rangle^n \subseteq P$, for some positive integer n . Conversely, assume that if $a\gamma(S\beta b) \subseteq P$ implies that $a \in P$ or $b^n \in P$ for some positive integer n , where $\gamma, \beta \in \Gamma$ and $a, b \in S$. Suppose that $A\Gamma B \subseteq$

P , where A and B are left Γ -ideals of S such that $A \not\subseteq P$. Then there exists $x \in A$ such that $x \notin P$. Now

$$x\gamma(S\beta\gamma) \subseteq A\Gamma(S\Gamma B) \subseteq A\Gamma B \subseteq P,$$

for all $\gamma \in B$. So by hypothesis, $\gamma^n \in P$ for all $\gamma \in B$. implies that $B^n \subseteq P$. Hence P is a quasi Γ -primary ideal in S .

Lemma 17. If S is a Γ -AG-groupoid with a left identity, then a left Γ -ideal P of S is a quasi Γ -primary if, and only if, $(S\gamma a)\delta(S\beta b) \subseteq P$ implies that $a \in P$ or $b^n \in P$ for some positive integer n , where $\gamma, \beta, \delta \in \Gamma$ and $a, b, \in S$.

Proof. Let P be a quasi Γ -primary ideal of a Γ -AG-groupoid S with left identity. Now suppose that $(S\gamma a)\delta(S\beta b) \subseteq P$. Then by definition of a left ideal, we get

$$\begin{aligned} (Saa)\delta(S\beta b) &= (S\gamma S)\delta(a\beta b) \\ &= S\delta(a\beta b) \\ &= a\delta(S\beta b) \end{aligned}$$

that is $a\alpha(S\beta b) = (S\gamma a)\delta(S\beta b) \subseteq P$. By Lemma 16, we have $a \in P$ or $b^n \in P$ for some positive integer n . Conversely, assume that if $(S\gamma a)\delta(S\beta b) \subseteq P$, then $a \in P$ or $b^n \in P$ for some positive integer n , where $\gamma, \beta, \delta \in \Gamma$ and $a, b \in S$. Let $a\delta(S\beta b) \subseteq P$. Now consider

$$a\delta(S\beta b) = (S\gamma a)\delta(S\beta b) \subseteq P.$$

By using the given assumption, if $a\delta(S\beta b) \subseteq P$, then $a \in P$ or $b^n \in P$ for some positive integer n . Then by Lemma 16, we have P is a quasi Γ -primary ideal in S .

Theorem 18. If S is a Γ -AG-groupoid with a left identity, then a left Γ -ideal P of S is a quasi Γ -primary if, and only if, $a\gamma b \in P$ implies that $a \in P$ or $b^n \in P$ for some positive integer n , where $\gamma \in \Gamma$ and $a, b, \in S$.

Proof. Let P be a left Γ -ideal of a Γ -AG-groupoid S with a left identity. Now suppose

that $a\gamma b \in P$. Then, by Ddefinition of a left ideal, we get

$$\begin{aligned} (Saa)\beta(S\gamma b) &= (SaS)\beta(a\gamma b) \\ &= S\beta(a\gamma b) \\ &\subseteq S\Gamma P \\ &\subseteq P. \end{aligned}$$

By Lemma 17, we have $a \in P$ or $b^n \in P$ for some positive integer n . Conversely, the proof is easy.

Theorem 19. Let S be a Γ -AG-groupoid, and let A be a quasi Γ -primary ideal of S . Then $(A:\gamma:r)$ is a quasi Γ -primary ideal in S , where $\gamma \in \Gamma$ and $r \in S$.

Proof. Assume that A is a quasi Γ -primary ideal of S . By Lemma 11, we have $(A:\gamma:r)$ is a left ideal in S . Let $a\beta b \in (A:\gamma:r)$. Suppose that $b^n \notin (A:\gamma:r)$, for all positive integers n . Since $a\beta b \in (A:\gamma:r)$, we have $r\gamma(a\beta b) \in A$ so that $a\gamma(r\beta b) \in A$. By Theorem 18, we have $a \in A \subseteq (A:\gamma:r)$ or $(r\beta b)^n \in A$, for some positive integer n . Therefore $a \in (A:\gamma:r)$, and hence $(A:\gamma:r)$, is a quasi Γ -primary ideal in S .

Theorem 20. Let S be a Γ -AG-groupoid with a left identity, and let P be a Γ -primary ideal of S . If $(S\gamma a^2)\alpha(S\beta b^2) \subseteq P$, then $a^2 \in P$ or $b^n \in P$, for some positive integer n .

Proof. Let P be a Γ -primary ideal of a Γ -AG-groupoid S with a left identity. Suppose that $b^n \notin P$, for all positive integers n . Now assume that $(S\gamma a^2)\alpha(S\beta b^2) \subseteq P$. Then by definition of a left ideal, we get

$$\begin{aligned} (S\gamma a^2)\alpha(S\beta b^2) &= ((S\beta b^2)\gamma a^2)\alpha S \\ &= ((a^2\beta b^2)\gamma S)\alpha S \\ &= (S\gamma S)\alpha(a^2\beta b^2) \\ &= a^2\alpha((S\gamma S)\beta b^2) \\ &= a^2\alpha((b^2\gamma S)\beta S) \\ &= (b^2\gamma S)\alpha(a^2\beta S) \end{aligned}$$

that is $(b^2\gamma S)\alpha(a^2\beta S) \subseteq P$. By Lemma 10, we have $a^2\beta S, b^2\gamma S$ are Γ -ideals in S so that

$$\begin{aligned}
 a^2 &= a\lambda a \\
 &= (e\chi a)\lambda a \\
 &= (a\chi a)\lambda e \\
 &= (a\lambda a)\beta e \\
 &= a^2\beta e \in a^2\beta S \subseteq P \\
 \text{or} \quad b^2 &= b\lambda b \\
 &= (e\chi b)\lambda b \\
 &= (b\chi b)\lambda e \\
 &= (b\chi b)\gamma e \\
 &= b^2\gamma e \in b^2\gamma S \subseteq P.
 \end{aligned}$$

Therefore, $a^2 \in P$.

Corollary 21. Let S be a Γ -AG-groupoid with a left identity, and let P be a Γ -primary ideal of S . If $b^2\gamma a^2 \in P$ then $a^2 \in P$ or $b^n \in P$, for some positive integer n .

Proof. Let P be a primary ideal of an AG-groupoid S with a left identity. Suppose that $b^n \notin P$, for all positive integers n . Now assume that $b^2\gamma a^2 \in P$. Then by definition of a left Γ -ideal, we get

$$\begin{aligned}
 (a^2\beta S)\alpha(b^2\gamma S) &= b^2\alpha((a^2\beta S)\gamma S) \\
 &= b^2\alpha((S\beta S)\gamma a^2) \\
 &= (S\beta S)\alpha(b^2\gamma a^2) \\
 &= S\alpha(b^2\gamma a^2) \\
 &\subseteq S\Gamma P \\
 &\subseteq P
 \end{aligned}$$

that is $(a^2\beta S)\alpha(b^2\gamma S) \subseteq P$. It is easy to see that $a^2 \in P$.

Definition 22. A Γ -AG-groupoid S is called a Γ -AG-3-band if its every element satisfies

$$a\alpha(a\beta a) = (a\alpha a)\beta a = a.$$

Proposition 23. Every left identity in a Γ -AG-3-band is a right identity.

Proof. Let e be a left identity and a be any element in a Γ -AG-3-band S . Then

$$\begin{aligned}
 a\gamma e &= (a\alpha(a\beta a))\gamma e \\
 &= (ea(a\beta a))\gamma a \\
 &= (a\beta a)\gamma a \\
 &= a
 \end{aligned}$$

for all $a \in S$ and $\gamma, \alpha, \beta \in \Gamma$. Hence e is a right identity.

Lemma 24. If a Γ -AG-3-band S has a left identity, then every left Γ -ideal is a Γ -ideal.

Proof. Let S be a Γ -AG-3-band and let A be a left Γ -ideal. Then

$$\begin{aligned}
 a\gamma S &= ((a\alpha a)\beta a)\gamma s \\
 &= (s\beta a)\gamma(a\alpha a) \\
 &= (S\Gamma A)\Gamma(A\Gamma A) \\
 &\subseteq A\Gamma A \\
 &\subseteq A
 \end{aligned}$$

for all $\gamma, \alpha, \beta \in \Gamma, a \in A$ and $s \in S$. Hence, A is a Γ -ideal.

Theorem 25. Let S be a Γ -AG-3-band with a left identity. Then, P is a quasi Γ -primary ideal in S if, and only if, S is a Γ -primary ideal in S .

Proof. The proof is straightforward.

Conclusions

Many new classes of Γ -AG-groupoids have been discovered recently. All this has attracted researchers of the field to investigate these newly discovered classes in detail. This current article investigates the Γ -ideals, Γ -primary, and quasi Γ -primary ideals in Γ -AG-groupoids. Some characterizations of Γ -primary ideals and quasi Γ -primary ideals are obtained. Moreover, the relationships between Γ -primary and quasi Γ -primary ideals in Γ -AG-groupoids have been investigated. Finally, the necessary and sufficient conditions of a Γ -primary ideal to be a quasi Γ -primary ideal in Γ -AG-groupoids have been obtained.

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