

Some Basic Properties of Primary and Left Primary Ideals in Γ -left Almost Rings

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Abstract

In this paper, we study primary and left primary ideals in Γ -LA-rings. Some characterizations of primary and left primary ideals are obtained. Moreover, we investigate relationships between primary and left primary ideals in Γ -LA-rings. Finally, we obtain necessary and sufficient conditions of a primary to be a left primary.

Mathematics Subject Classification: 16A76, 20M25, 20N02

Keywords: Γ -LA-ring, primary ideal, left primary ideal, left (right) ideal

1 Introduction

Abel-Grassmann's groupoid (AG-groupoid) is the generalization of semigroup theory with the wide range of usages in theory of flocks [5]. The fundamentals of this non-associative algebraic structure were the first discovered by Kazim

and Naseeruddin [1]. AG-groupoid is a non-associative algebraic structure mid way between a groupoid and a commutative semigroup. It is interesting to note that an AG-groupoid with right identity becomes a commutative monoid [3]. This structure is closely related with a commutative semigroup because if an AG-groupoid contains a right identity, then it becomes a commutative monoid [3]. A left identity in an AG-groupoid is unique. Ideals in AG-groupoids have been discussed by Mushtaq and Yousuf [4, 3].

In 1981, the notion of Γ -semigroups was introduced by Sen. Let S and Γ be any nonempty sets. If there exists a mapping $S \times \Gamma \times S \rightarrow S$ written (a, α, c) by $a\alpha c$, S is called a Γ -semigroup if S satisfies the identity:

$$(a\alpha b)\beta c = a\alpha(b\beta c)$$

for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. A Γ -AG-groupoids analogous to Γ -semigroups. A groupoid S is called a Γ -AG-groupoid if it satisfies the left invertive law:

$$(a\gamma b)\delta c = (c\gamma b)\delta a$$

for all $a, b, c, d \in S$ and $\gamma, \delta \in \Gamma$ [2]. This structure is also known as left almost semigroup(LA-semigroup).

S.M. Yusuf in [15] introduces the concept of a left almost ring (LA-ring). That is, a non-empty set R with two binary operations $+$ and $\cdot$ is called a left almost ring, if $(R, +)$ is an LA-group, $(R, \cdot)$ is an LA-semigroup and distributive laws of $\cdot$ over $+$ holds. Further in [10] T. Shah and I. Rehman generalize the notions of commutative semigroup rings into LA-semigroup LA-rings. However T. Shah and Fazal ur Rehman in [10] generalize the notion of an LA-ring into an nLA-ring. A near left almost ring (nLA-ring) N is an LA-group under $+$, an LA-semigroup under $\cdot$ and left distributive property of $\cdot$ over $+$ holds.

T. Shah, Fazal ur Rehman and M. Raees asserted that a commutative ring $(R, +, \cdot)$, we can always obtain an LA-ring $(R, \oplus, \cdot)$ by defining, for $a, b, c \in R$, $a \oplus b = ba$ and $a \cdot b$ is same as in the ring. In this paper, we study primary and left primary ideals in Γ -LA-rings. Some characterizations of primary and left primary ideals are obtained. Moreover, we investigate relationships between primary and left primary.

2 Primary and Left Primary ideals in Γ -LA-rings

We start with the following theorem that gives a relation between primary and left primary ideal in Γ -LA-ring. Our starting points is the following definition:

Definition 2.1. An ideal P is called primary if $A\Gamma B \subseteq P$ implies that $A \subseteq P$ or $B^n \subseteq P$, for some positive integer n , where A, B are ideals of R .

Definition 2.2. A left ideal P is called left primary if $A\Gamma B \subseteq P$ implies that $A \subseteq P$ or $B^n \subseteq P$, for some positive integer n , where A, B are a left ideals of R .

Remark 2.3. It is easy to see that every left primary ideal is primary.

Lemma 2.4. If R is a Γ -LA-ring with left identity, then a left ideal P of R is left primary if and only if $a\gamma(R\alpha b) \subseteq P$ implies that $a \in P$ or $b^n \in P$ for some positive integer n , where $\gamma, \alpha \in \Gamma$ and $a, b \in R$.

Proof. Let P be a left primary ideal of a Γ -LA-ring R with left identity. Now suppose that $a\gamma(R\alpha b) \subseteq P$. Then by Definition of left ideal, we get $R\beta(a\gamma(R\alpha b)) \subseteq R\beta P \subseteq P$ that is

$$\begin{aligned} R\beta(a\gamma(R\alpha b)) &= (R\delta R)\beta(a\gamma(R\alpha b)) \\ &= (R\delta a)\beta(R\gamma(R\alpha b)) \\ &= (R\delta a)\beta((R\delta R)\gamma(R\alpha b)) \\ &= (R\delta a)\beta((b\delta R)\gamma(R\alpha R)) \\ &= (R\delta a)\beta((b\delta R)\gamma R) \\ &= (R\delta a)\beta((R\delta R)\gamma b) \\ &= (R\delta a)\beta(R\gamma b) \end{aligned}$$

for all $\gamma, \alpha, \beta, \delta \in \Gamma$. Which implies that $(R\delta a)\beta(R\gamma b) \subseteq P$ so $a = e\delta a \in R\delta a \subseteq P$ or $b^n = (e\gamma b)^n \in (R\gamma b)^n$, for some positive integer n . Conversely, assume that if $a\gamma(R\alpha b) \subseteq P$ implies that $a \in P$ or $b^n \in P$ for some positive integer n , where $\gamma, \alpha \in \Gamma$ and $a, b \in R$. Suppose that $A\Gamma B \subseteq P$, where A and B are left ideals of R such that $A \not\subseteq P$. Then there exists $x \notin A$ such that $x \in P$. Now

$$x\gamma(R\alpha y) \subseteq A\Gamma(R\Gamma B) \subseteq A\Gamma B \subseteq P,$$

for all $y \in B$. So by hypothesis, $y^n \in P$ for all $y \in B$ implies that $B^n \subseteq P$. Hence P is left primary ideal in R . $\square$

Lemma 2.5. If R is a Γ -LA-ring with left identity, then a left ideal P of R is left primary if and only if $(R\gamma a)\beta(R\alpha b) \subseteq P$ implies that $a \in P$ or $b^n \in P$ for some positive integer n , where $\gamma, \alpha, \beta \in \Gamma$ and $a, b \in R$.

Proof. Let R be a Γ -LA-ring with left identity and let P be a left primary of R . Now suppose that $(R\gamma a)\beta(R\alpha b) \subseteq P$. Then by Definition of left ideal, we get

$$\begin{aligned} (R\gamma a)\beta(R\alpha b) &= (R\gamma R)\beta(a\alpha b) \\ &= R\beta(a\alpha b) \\ &= a\beta(R\alpha b) \end{aligned}$$

that is $a\beta(R\alpha b) \subseteq P$. By Lemma 2.4, we have $a \in P$ or $b^n \in P$, for some positive integer n . Conversely, assume that if $(R\gamma a)\beta(R\alpha b) \subseteq P$ implies that

$a \in P$ or $b^n \in P$ for some positive integer n , where $\gamma, \alpha, \beta \in \Gamma$ and $a, b \in R$. Let $a\gamma(R\alpha b) \subseteq P$. Now consider

$$a\gamma(R\alpha b) = (R\gamma a)\beta(R\alpha b)$$

By using given assumption, if $a\gamma(R\alpha b) \subseteq P$, then $a \in P$ or $b^n \in P$ for some positive integer n . Then by Lemma 2.4, we have P is a left primary ideal in R . $\square$

Corollary 2.6. *Let R be a Γ -LA-ring with left identity and let P be a primary of R . If $(R\gamma a^2)\beta(R\alpha b^2) \subseteq P$, then $a^2 \in P$ or $b^n \in P$ for some positive integer n , where $\gamma, \alpha, \beta \in \Gamma$ and $a, b \in R$.*

Proof. This follows from Lemma 2.5. $\square$

Theorem 2.7. *If R is a Γ -LA-ring with left identity, then a left ideal P of R is left primary if and only if $a\gamma b \in P$ implies that $a \in P$ or $b^n \in P$ for some positive integer n , where $\gamma \in \Gamma$ and $a, b \in R$.*

Proof. Let P be a left ideal of Γ -LA-ring R with left identity. Now suppose that $a\gamma b \in P$. Then by Definition of left ideal, we get

$$\begin{aligned} (R\gamma a)\beta(R\alpha b) &= (R\gamma R)\beta(a\alpha b) \\ &= R\beta(a\alpha b) \\ &\subseteq R\beta P \\ &\subseteq P. \end{aligned}$$

By Lemma 2.5, we have $a \in P$ or $b^n \in P$ for some positive integer n . Conversely, the proof is easy. $\square$

Corollary 2.8. *Let R be a Γ -LA-ring with left identity and let P be a primary of R . If $a^2\gamma b^2 \in P$, then $a^2 \in P$ or $b^n \in P$ for some positive integer n , where $\gamma \in \Gamma$ and $a, b \in R$.*

Proof. This follows from Theorem 2.7. $\square$

Theorem 2.9. *Let R be a Γ -LA-ring with left identity, and let A be a left primary ideal of R . Then $(A : \gamma : r)$ is a left primary ideal in R , where $\gamma \in \Gamma$ and $r \in R$.*

Proof. Assume that A is a left primary ideal of R . By Corollary ??, we have $(A : \gamma : r)$ is a left ideal in R . Let $a\alpha b \in (A : \gamma : r)$. Suppose that $b^n \notin (A : \gamma : r)$, for all positive integer n . Since $a\alpha b \in (A : \gamma : r)$, we have $r\gamma(a\alpha b) \in A$ so that $a\gamma(r\alpha b) \in A$. By Theorem 2.7, we have $a \in A$ or $(r\alpha b)^n \in A$, for some positive integer n . Therefore $a \in (A : \gamma : r)$ and hence $(A : \gamma : r)$ is a left primary ideal in R . $\square$

Corollary 2.10. *Let R be a Γ -LA-ring with left identity, and let A be a left primary ideal of R . Then $(A : \Gamma : B)$ is a left primary ideal in R , where $B \subseteq R$.*

Proof. This follows from Theorem 2.9. $\square$

Definition 2.11. *A Γ -LA-ring R is called Γ -LA-3-band if its every element satisfies $(a\alpha a)\beta a = a\alpha(a\beta a) = a$.*

Theorem 2.12. *Every left identity in a Γ -LA-3-band is a right identity.*

Proof. Let e be a left identity and a be any element in a Γ -LA-3-band R . Then

$$\begin{aligned} a\alpha e &= (a\delta(a\beta a))\alpha e \\ &= (e\delta(a\beta a))\alpha a \\ &= (a\beta a)\alpha a \\ &= a \end{aligned}$$

for all $\alpha, \beta, \delta \in \Gamma$. Hence e is right identity. $\square$

Lemma 2.13. *If a Γ -LA-3-band R has a left identity, then every left ideal is a ideal.*

Proof. Let R be a Γ -LA-3-band and let A be a left ideal. Then

$$\begin{aligned} a\gamma s &= ((a\alpha a)\beta a)\gamma s \\ &= (s\beta a)\gamma(a\alpha a) \\ &\in (R\Gamma A)\Gamma(A\Gamma A) \\ &\subseteq A\Gamma A \\ &\subseteq A \end{aligned}$$

for all $\gamma, \alpha, \beta \in \Gamma, a \in A$ and $s \in R$. Hence A is a ideal. $\square$

Theorem 2.14. *Let R be a Γ -LA-3-band with left identity. Then P is a left primary ideal in R if and only if P is a primary ideal in R .*

Proof. The proof is straightforward. $\square$

Acknowledgements. The authors are very grateful to the anonymous referee for stimulating comments and improving presentation of the paper.

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Received: January 27, 2015; Published: March 9, 2015