

On Quasi-Left Primary Γ -ideals in Γ -AG-Groupoids

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Abstract

The purpose of this paper is to introduce the notion of a quasi-left primary Γ -ideals in Γ -AG-groupoids, we study left primary and quasi-left primary Γ -ideals in Γ -AG-groupoids. Some characterizations of left primary and quasi-left primary Γ -ideals are obtained. Moreover, we investigate the relationships between left primary and quasi left-primary ideals in Γ -AG-groupoids.

Keywords: Γ -AG-groupoid, Γ -LA-semigroup, Γ -ideal, quasi-left primary, left primary.

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INTRODUCTION

Abel-Grassmann's groupoid (AG-groupoid) is a generalization of semigroup theory with wide range of usages in theory of flocks (Naseeruddin, 1970). The fundamentals of this non-associative algebraic structure were first discovered by Kazim and Naseeruddin (1972). A groupoid S is called an AG-groupoid if it satisfies the left invertive law:

$$(ab)c = (cb)a$$

for all $a, b, c \in S$. It is interesting to note that an AG-groupoid with right identity becomes a commutative monoid (Mushtaq and Yousuf, 1978). This structure is closely related to a commutative semigroup. Because of containing a right identity, an AG-groupoid becomes a commutative monoid (Mushtaq and Yousuf, 1978). A left identity in an AG-groupoid is unique (Mushtaq and Yousuf, 1978). It lies between a groupoid and a commutative semigroup with wide range of applications in theory of flocks (Naseeruddin, 1970). Ideals in AG-groupoids have been discussed in (Mushtaq and Yousuf, 1988). In 1981, the notion of Γ -semigroups was introduced by M. K. Sen. A groupoid S is called a Γ -AG-groupoid if it satisfies the left invertive law:

$$(a\gamma b)\delta c = (c\gamma b)\delta a$$

for all $a, b, c \in S$ and $\gamma, \delta \in \Gamma$ (Khan, 2013). This structure is also known as a left almost semigroup (LA-semigroup). In this paper, we are going to investigate some interesting properties of recently discovered classes, namely Γ -AG-groupoid S always satisfies the Γ -medial law:

$$(a\gamma b)\beta(c\delta d) = (a\gamma c)\beta(b\delta d)$$



for all $a, b, c, d \in S$ and $\gamma, \beta, \delta \in \Gamma$ (Khan, 2013), while a Γ -AG-groupoid S with left identity e always satisfies Γ -paramedial law:

$$(\alpha\gamma b)\beta(c\delta d) = (d\gamma c)\beta(b\delta a)$$

for all $a, b, c, d \in S$ and $\gamma, \beta, \delta \in \Gamma$ (Khan, 2013). Recently T. Shah and I. Rehman have discussed Γ -Ideals and Γ -Bi-Ideals in Γ -AG-Groupoids.

In this paper we characterize the Γ -AG-groupoid. We investigate the relationships between left primary and quasi- left primary Γ -ideals in Γ -AG-groupoids.

BASIC RESULTS

In this section, we refer to the article of Shah and Rehman in 2010, for some elementary aspects and quote few definitions, and essential examples to step up this study. For more details, we refer to the papers in the references.

Definition 2.1. (Shah and Rehman, 2010) A sub Γ -AG-groupoid A of S is called a left (right) Γ -ideal of S if $S\Gamma A \subseteq A$ ($A\Gamma S \subseteq A$) and is called an Γ -ideal if it is left as well as right Γ -ideal.

Lemma 7. (Shah and Rehman, 2010) If A is a left Γ -ideal of a Γ -AG-groupoid S with left identity, and if for any $a \in S$, there exists $\gamma \in \Gamma$, then $a\gamma A$ is a left Γ -ideal of S .

Lemma 9. If S is a Γ -AG-groupoid with left identity e , then $a\gamma b = a\beta b$ for all $a, b \in S$ and $\gamma, \beta \in \Gamma$.

Proof. Let S be a Γ -AG-groupoid with left identity e , and let $a, b \in S$ and $\gamma, \beta \in \Gamma$ therefore we have

$$\begin{aligned} a\gamma b &= a\gamma(e\beta b) \\ &= e\gamma(a\beta b) \\ &= a\beta b. \end{aligned}$$

Hence $a\gamma b = a\beta b$. □

MAIN RESULTS

We start with the following theorem that gives a relation between left primary and quasi- left primary Γ -ideal in Γ -AG-groupoid. Our starting points are the following definitions:

Definition 3.1. A left Γ -ideal P is called left primary if $A\Gamma B \subseteq P$ implies that $A \subseteq P$ or $((B\Gamma B)\Gamma B)\Gamma \dots \Gamma B = B^n \subseteq P$, for some positive integer n , where A and B are two left ideals of S .

Definition 3.2. A left Γ -ideal P is called quasi-left primary if $A\Gamma B \subseteq P$ implies that $((A\Gamma A)\Gamma A)\Gamma \dots \Gamma A = A^n \subseteq P$ or $((B\Gamma B)\Gamma B)\Gamma \dots \Gamma B = B^n \subseteq P$, for some positive integer n , where A and B are two left Γ -ideals of S .

Remark. It is easy to see that every left primary Γ -ideal is quasi-left primary.

Lemma 3.3. If S is a Γ -AG-groupoid with left identity, then a left Γ -ideal P of S is quasi-left primary if and only if $a\gamma(S\beta b) \subseteq P$ implies that

$$(((a\delta a)\delta a)\delta \dots)\delta a = a^n \in P \text{ or } (((b\delta b)\delta b)\delta \dots)\delta b = b^n \in P$$

for some positive integer n , where $\gamma, \beta, \delta \in \Gamma$ and $a, b \in S$.

Proof. Let P be a quasi left primary left Γ -ideal of a Γ -AG-groupoid S with left identity. Now suppose that $a\gamma(S\beta b) \subseteq P$. Then by Definition of left Γ -ideal, we get $ST(a\gamma(S\beta b)) \subseteq STP \subseteq P$ that is,

$$\begin{aligned} ST(a\gamma(S\beta b)) &= (S\delta S)\Gamma(a\gamma(S\beta b)) \\ &= (S\delta a)\Gamma(S\gamma(S\beta b)) \\ &= (S\delta a)\Gamma((STS)\gamma(S\beta b)) \\ &= (S\delta a)\Gamma((b\beta S)\gamma(STS)) \\ &= (S\delta a)\Gamma((b\beta S)\gamma(S)) \\ &= (S\delta a)\Gamma((S\beta S)\gamma b) \\ &= (S\delta a)\Gamma(S\gamma b) \end{aligned}$$

for all $\delta \in \Gamma$. Since $ST(a\gamma(S\beta b)) \subseteq P$ and $ST(a\gamma(S\beta b)) = (S\delta a)\Gamma(S\gamma b)$, we have $(S\delta a)\Gamma(S\gamma b) \subseteq P$ so that $a^n = (e\delta a)^n \in (S\delta a)^n \subseteq P$ or $b^n = (eb)^n \in (S\gamma b)^n \subseteq P$, for some positive integer n . Conversely, assume that if $a\gamma(S\beta b) \subseteq P$ implies that $a^n \in P$ or $b^n \in P$ for some positive integer n , where $\gamma, \beta \in \Gamma$ and $a, b \in S$. Suppose that $A\Gamma B \subseteq P$, where A and B are left Γ -ideals of S such that $A^n \not\subseteq P$, for all positive integer n . Then there exists $x \in A^n$ such that $x \notin P$. Now

$$x\gamma(S\beta y) \subseteq A\Gamma(STB) \subseteq A\Gamma B \subseteq P,$$

for all $y \in B$. So by hypothesis, $y^n \in P$, for all $y \in B$ implies that $B^n \subseteq P$. Hence P is quasi-left primary ideal in S . $\square$

Lemma 3.4. If S is a Γ -AG-groupoid with left identity, then a left Γ -ideal P of S is quasi left primary if and only if $(S\gamma a)\delta(S\beta b) \subseteq P$ implies that $a^n \in P$ or $b^n \in P$ for some positive integer n , where $\gamma, \beta, \delta \in \Gamma$ and $a, b \in S$.

Proof. Let P be a quasi-left primary Γ -ideal of a Γ -AG-groupoid S with left identity. Now suppose that $(S\gamma a)\delta(S\beta b) \subseteq P$. Then by Definition of left Γ -ideal, we get

$$\begin{aligned} (S\gamma a)\delta(S\beta b) &= (S\gamma S)\delta(a\beta b) \\ &= S\delta(a\beta b) \\ &= a\delta(S\beta b) \end{aligned}$$

that is $a\delta(S\beta b) = (S\gamma a)\delta(S\beta b) \subseteq P$. By Lemma 3.3, we have $a^n \in P$ or $b^n \in P$ for some positive integer n . Conversely, assume that if $(S\gamma a)\delta(S\beta b) \subseteq P$, then $a^n \in P$ or $b^n \in P$ for some positive integer n , where $\gamma, \beta, \delta \in \Gamma$ and $a, b \in S$. Let $a\delta(S\beta b) \subseteq P$. Now consider



$$a\delta(S\beta b) = (S\gamma a)\delta(S\beta b) \subseteq P.$$

By using given assumption, if $a\delta(S\beta b) \subseteq P$, then $a^n \in P$ or $b^n \in P$ for some positive integer n .

Then by Lemma 3.3, we have P is a quasi-left primary Γ -ideal in S . $\square$

Theorem 3.5. If S is a Γ -AG-groupoid with left identity, then a left Γ -ideal P of S is quasi-left primary if and only if $a\gamma b \in P$ implies that $a^n \in P$ or $b^n \in P$ for some positive integer n , where $\gamma \in \Gamma$ and $a, b \in S$.

Proof. Let P be a left Γ -ideal of a Γ -AG-groupoid S with left identity. Now suppose that $a\gamma b \in P$. Then by Definition of left ideal, we get

$$\begin{aligned} (S\alpha a)\beta(S\gamma b) &= (S\alpha S)\beta(a\gamma b) \\ &= S\beta(a\gamma b) \\ &\subseteq S\Gamma P \\ &\subseteq P. \end{aligned}$$

By Lemma 3.4, we have $a^n \in P$ or $b^n \in P$ for some positive integer n .

Conversely, the proof is easy. $\square$

Theorem 3.6. Let S be a Γ -AG-groupoid with left identity and let P be a semiprime left ideal of S . Then P is a quasi-left primary Γ -ideal in S if and only if S is a primary Γ -ideal in S .

Proof. The proof is straightforward. $\square$

CONCLUSIONS

Many new classes of Γ -AG-groupoids have been discovered recently. All these have attracted researchers of the field to investigate these newly discovered classes in detail. This article investigates the Γ -ideals, left primary and quasi-left primary Γ -ideals in Γ -AG-groupoids. Some characterizations of left primary ideal and quasi-left primary ideals are obtained. Moreover, we investigate the relationships between left primary and quasi left-primary ideals in Γ -AG-groupoids.

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