



On Filters and Upper sets in Γ -CI-Algebras

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Abstract

The purpose of this paper is to introduce the notion of a filter and upper sets in Γ -CI-algebras, we study transitive, filter and upper sets in Γ -CI-algebras. Some characterizations of filter and upper sets are obtained.

Keywords: CI-algebra, Γ -CI-algebra, upper set, filter, transitive.

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INTRODUCTION

Let X denote an CI-algebra unless otherwise specified. By an CI-algebra we mean an algebra $(X; *, 1)$ of type $(2, 0)$ with a single binary operation " $*$ " that satisfies the following identities: for any $x, y, z \in X$,

1. $x * x = 1$,
2. $1 * x = x$,
3. $x * (y * z) = y * (x * z)$

In 2006, H. S. Kim and Y. H. Kim defined a BE-algebra. Biao Long Meng, defined notion of CI-algebra as a generalization of a BE-algebra (Meng B.L., 2010). BE-algebras and CI-algebras are studied in detail by some researchers (Borumand Saeid A. & Rezaei A., 2012; Kim K.H., 2011; Meng B.L., 2010; Piekart B. & Andrzej Walendziak A., 2011) and some fundamental properties of CI-algebra are discussed. It is easy to see that every CI-algebras is Γ -CI-algebras.

Our aim in this paper is to introduce the notion of a Γ -CI-algebras, we study transitive, filter and upper sets in Γ -CI-algebras. Some characterizations of filter and upper sets are obtained.

BASIC RESULTS

In this section is to introduce the notion of a Γ -CI-algebras.

Definition 2.1. Let X and Γ be any nonempty sets. The structure $(\Gamma, X; 1)$ is called a Γ -CI-algebra if there exists a mapping $X \times \Gamma \times X \rightarrow X$ written as (x, γ, y) by $x\gamma y$, that satisfies the following condition

1. $x\gamma x = 1$ for all $x \in X$ and $\gamma \in \Gamma$,
2. $1\gamma x = x$ for all $x \in X$ and $\gamma \in \Gamma$,
3. $x\gamma(y\beta z) = y\gamma(x\beta z)$

for all $x, y, z \in X$ and $\gamma, \beta \in \Gamma$.

Throughout this paper X will denote a Γ -CI-algebra. We introduce a relation $\leq$ on X by $x \leq y$ if and only if $x\gamma y = 1$.

Example 2.2. Let $(\Gamma, X; 1)$ be an arbitrary CI-algebra and Γ any nonempty set. Define a mapping $X \times \Gamma \times X \rightarrow X$, by $x\gamma y = x * y$ for all $x, y \in X$ and $\gamma \in \Gamma$. It is easy to see that X is a Γ -CI-algebra. Indeed,

$$\begin{aligned} 1. \quad x\gamma x &= x * x \\ &= 1, \\ 2. \quad 1\gamma x &= 1 * x \\ &= x, \\ 3. \quad x\gamma(y\beta z) &= x * (y\beta z) \\ &= x * (y * z) \\ &= y * (x * z) \\ &= y\gamma(x\beta z) \end{aligned}$$

for all $x, y, z \in X$ and $\gamma, \beta \in \Gamma$. Thus every CI-algebra implies a Γ -CI-algebra.

Example 2.3. Let $X = \{1, a, b\}$ in which “ $\cdot$ ” is defined by

$\cdot$	1	a	b
1	1	b	a
a	a	1	b
b	b	a	1

Let $\Gamma \neq \emptyset$. Define a mapping $X \times \Gamma \times X \rightarrow X$ by $x\gamma y = y \cdot x$ for all $x, y \in X$ and $\gamma \in \Gamma$. Then X is a Γ -CI-algebra. Indeed

$$\begin{aligned} 1. \quad x\gamma x &= (x \cdot x) \cdot (x \cdot x) \\ &= 1 \cdot 1 \\ &= 1, \\ 2. \quad 1\gamma x &= x \cdot 1 \\ &= x, \\ 3. \quad x\gamma(y\beta z) &= (y\beta z) \cdot x \end{aligned}$$



$$\begin{aligned}
 &= (z \cdot y) \cdot x \\
 &= (z \cdot x) \cdot y \\
 &= (x\beta z) \cdot y \\
 &= y\gamma(x\beta z)
 \end{aligned}$$

for all $x, y, z \in X$ and $\gamma, \beta \in \Gamma$. Therefore X is a Γ -CI-algebra. Then $(\Gamma, X, 1)$ is a Γ -CI-algebra, which is not a CI-algebra. For any $x_1, \dots, x_n, a \in X$ and $\gamma_1, \dots, \gamma_n \in \Gamma$ we define

$$\prod_{i=1}^n x_i \gamma_i a = x_n \gamma_n (\dots (x_1 \gamma_1 a) \dots)$$

Lemma 2.4. Let X be a Γ -CI-algebra. Then $x\gamma y = x\beta y$ for any $x, y \in X$ and $\gamma, \beta \in \Gamma$.

Proof. Let $x, y \in X$ and let $\gamma, \beta \in \Gamma$. Then

$$\begin{aligned}
 x\gamma y &= x\gamma(1\beta y) \\
 &= 1\gamma(x\beta y) \\
 &= x\beta y.
 \end{aligned}$$

Hence $x\gamma y = x\beta y$. □

Proposition 2.5. Any Γ -CI-algebra X satisfies for any $x, y \in X$ and $\gamma, \beta, \alpha \in \Gamma$, $y\gamma[(y\alpha x)\beta\alpha] = 1$

Proof. Let $x, y \in X$ and $\gamma, \beta, \alpha \in \Gamma$. Then

$$\begin{aligned}
 y\gamma[(y\alpha x)\beta\alpha] &= (y\alpha x)\gamma(y\beta\alpha) \\
 &= (y\beta x)\gamma(y\beta\alpha) \\
 &= 1
 \end{aligned}$$

Hence $y\gamma[(y\alpha x)\beta\alpha] = 1$ □

Proposition 2.6. Let $x \in X$ and $\gamma \in \Gamma$ if $1 \leq x$, then $x = 1$

Proof. The proof is straightforward. □

Definition 2.7. A Γ -CI-algebra X is said to be transitive if for all $x, y, z \in X$ and $\gamma, \alpha, \beta \in \Gamma$, $y\gamma z \leq (x\alpha y)\gamma(x\beta z)$.

Lemma 2.8. If a Γ -CI-algebra X is transitive, then for all $x, y, z \in X$ and $\gamma \in \Gamma$, $x \leq y$ implies $z\gamma x \leq z\gamma y$.

Proof. The proof is straightforward. □

Theorem 2.9. Let X be a transitive Γ -CI-algebra and let $x, y \in X$ such that $x \leq y$. Then for all

$$a_1, \dots, a_n \in X, \gamma_1, \dots, \gamma_n \in \Gamma, \prod_{i=1}^n a_i \gamma_i x = 1 \text{ implies that } \prod_{i=1}^n a_i \gamma_i y = 1$$

Proof. We have $x \leq y$ and from Lemma 2.8 we see that

$$1 = \prod_{i=1}^n a_i \gamma_i x \leq \prod_{i=1}^n a_i \gamma_i y$$

Applying Proposition 2.6 we conclude that $\prod_{i=1}^n a_i \gamma_i y = 1$

□

FILTERS

The results of the following lemmas seem play an important role to study Filters in Γ -CI-algebra; these facts will be used so frequently that normally we shall make no reference to this definition.

Definition 3.1. A filter of X is a subset F of X such that for all $x, y \in X$ and $\gamma \in \Gamma$:

1. $1 \in F$
2. if $x\gamma y \in F$ and $x \in F$, then $y \in F$.

By $Fil(X)$ we denote the set of all filters in X . It is obvious that $\{1\}, X \in Fil(X)$

Proposition 3.2. If $F_i (i \in I)$ are filters of X , then $\bigcap_{i \in I} F_i$ is a filter of X .

Proof. The proof is straightforward.

□

Theorem 3.3. Let F be a subset of X containing 1. Then $F \in Fil(X)$ if and only if for any $a, b \in F, x \in X$ and $\gamma, \beta \in \Gamma, a\gamma(b\beta x) = 1$ implies $x \in F$.

Proof. $\Rightarrow$ Let $F \in Fil(X)$. Assume $a, b \in F, x \in X$ and $\gamma, \beta \in \Gamma$ such that $a\gamma(b\beta x) = 1$. From Definition 3.1 (1) we obtain $a\gamma(b\beta x) \in F$. Applying Definition 3.1 (2) twice we have $x \in F$.

$\Leftarrow$ Since $1 \in F$, the condition Definition 3.1 (1) holds. Suppose that $a\gamma x \in F$ and $a \in F$, for all $\gamma \in \Gamma$. By Proposition 2.5, $a\delta[(a\gamma x)\beta x] = 1$. Then $x \in F$ and hence Definition 3.1 (2) is true. Therefore $F \in Fil(X)$.

By induction we easily obtain

Corollary 3.4. Let F be a subset of X containing 1. Then $F \in Fil(X)$ if and only if for any

$a_i \in F, x \in X$ and $\gamma_i \in \Gamma, \prod_{i=1}^n a_i \gamma_i x = 1$ implies $x \in F$.

UPPER SETS

The results of the following lemmas seem play an important role to study upper set in Γ -CI-algebra; these facts will be used so frequently that normally we shall make no reference to this definitions.



Definition 4.1. Let X be a Γ -CI-algebra and $a \in X, \gamma \in \Gamma$. Define $A(a\gamma)$ by

$$A(a\gamma) = \{1\} \cup \{x \in X : a\gamma x = 1\}$$

Then we call $A(a\gamma)$ the initial section of the element a .

Definition 4.2. Let X be a Γ -CI-algebra and $a, b \in X, \gamma, \beta \in \Gamma$. Define $A(a\gamma, b\beta)$ by

$$A(a\gamma, b\beta) = \{1\} \cup \{x \in X : a\gamma(b\beta x) = 1\}$$

We call $A(a\gamma, b\beta)$ an upper set of a and b

Applying Definition 2.1 (2) we conclude that $A(\gamma x) = A(1, x)$.

Remark. 1. By Definition 4.2, we have $A(a\gamma, b\beta) = A(b\gamma, a\beta)$ for all $a, b \in X$ and $\gamma, \beta \in \Gamma$.

2. In general, not every filter is an upper set and not every upper set is a filter.

Theorem 4.3. Let X be a Γ -CI-algebra, $x, y \in X$ and $\gamma, \beta \in \Gamma$. Then

1. $1 \in A(a\gamma)$, $1 \in A(a\gamma, b\beta)$,
2. $a \in A(a\gamma)$,
3. if $y\beta 1 = 1$, then $A(a\gamma) \subseteq A(a\gamma, b\beta)$,
4. if $y\beta 1 \neq 1$, then $A(a\gamma) - \{1\} \subseteq X - A(a\gamma, b\beta)$.

Proof. Let X be a Γ -CI-algebra X , $a, b \in X$ and let $\gamma, \beta \in \Gamma$.

(1) By Definition 4.1 and Definition 4.1, we have $1 \in A(a\gamma)$ and $1 \in A(a\gamma, b\beta)$.

(2) Since $a\gamma a = 1$, we have $a \in A(a\gamma)$.

(3) Let $x \in A(a\gamma)$. Then $x \in \{x \in X : a\gamma x = 1\}$ or $x = \{1\}$

Case 1: $x = 1$. It is easy to see that $x \in A(a\gamma, b\beta)$.

Case 2: $x \in \{x \in X : a\gamma x = 1\}$. It is easy to see that $a\gamma x = 1$. Then

$$\begin{aligned} a\gamma(b\beta x) &= b\gamma(a\beta x) \\ &= b\gamma(a\gamma x) \\ &= b\gamma 1 \\ &= 1. \end{aligned}$$

Hence $x \in A(a\gamma, b\beta)$.

(4) Suppose that $y\beta 1 \neq 1$. Let $x \in A(a\gamma) - \{1\}$. Then $a\gamma x = 1$ and $x \neq 1$. We have

$$\begin{aligned} a\gamma(b\beta x) &= b\gamma(a\beta x) \\ &= b\gamma(a\gamma x) \\ &= b\gamma 1 \\ &\neq 1. \end{aligned}$$

Thus $x \notin A(a\gamma, b\beta)$ and hence $A(a\gamma) - \{1\} \subseteq X - A(a\gamma, b\beta)$. □

CONCLUSIONS

Many new classes of Γ -CI-algebras have been discovered recently. All these have attracted researchers of the field to investigate these newly discovered classes in detail. This article investigates we study filters and upper sets in Γ -CI-algebras. Some characterizations of filters and upper sets are obtained.

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