

On the Classical Primary Radical Formula and Classical Primary Subsemimodules

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ABSTRACT— In this paper, we characterize the classical primary radical of subsemimodules and classical primary subsemimodules of semimodules over a commutative semirings. Furthermore we prove that if N_j is a classical primary subsemimodule of M_j , then N_j is to satisfy the classical primary radical formula in M_j if and only if $M_1 \times M_2 \times \dots \times M_{j-1} \times N_j \times M_{j+1} \times \dots \times M_n$ is to satisfy the classical primary radical formula in M .

Keywords— classical primary subsemimodule, primary subsemimodule, classical primary radical, classical primary radical formula.

1. INTRODUCTION

Throughout this paper a semiring will be defined as follows: A semiring is a set R together with two binary operations called addition "+" and multiplication "·" such that $(R, +)$ is a commutative semigroup and $(R, \cdot)$ is semigroup; connecting the two algebraic structures are the distributive laws : $a(b+c) = ab+ac$ and $(a+b)c = ac+bc$ for all $a, b, c \in R$. A semimodule M over a semiring R is a commutative monoid M with additive identity 0, together with a function $R \times M \rightarrow M$, defined by $(r, m) \mapsto rm$ such that:

1. $r(m+n) = rm + rn$
2. $(r+s)m = rm + sm$
3. $(rs)m = r(sm)$
4. $r0 = 0 = 0m$
5. $1m = m$

for all $m, n \in M$ and $r, s \in R$. Clearly every ring is a semiring and hence every module over a ring R is a left semimodule over a semiring R . A nonempty subset N of a R -semimodule M is called subsemimodule of M if N is closed under addition and closed under scalar multiplication.

J. Saffar Ardabili S. Motmaen and A. Yousefian Darani in (2011) defined a different class of subsemimodules and called it classical prime. A proper subsemimodule N of M is said to be classical prime when for $a, b \in R$ and $m \in M, abm \in N$ implies that $am \in N$ or $bm \in N$.

Corollary 2.2. Let $N = \prod_{i=1}^n N_i$ be a subsemimodule of M . Then $\langle E_M(N) \rangle = \prod_{i=1}^n \langle E_{M_i}(N_i) \rangle$.

Proof. This follows from Proposition 2.1

Proposition 2.3. If N is a classical prime subsemimodule of M , then $\langle E_M(N) \rangle = N$.

Proof. Clearly, $N \subseteq \langle E_M(N) \rangle$. To show that $\langle E_M(N) \rangle \subseteq N$. Let $x = \sum_{i=1}^k r_i m_i \in \langle E_M(N) \rangle$, where $r_i^{k_i} m_i \in N$ for some $k_i \in \mathbb{N}^+$. Since N is a classical prime subsemimodule of M , we have $r_i m_i \in N$. Then $x = \sum_{i=1}^k r_i m_i \in N$ so that $\langle E_M(N) \rangle \subseteq N$. Hence $\langle E_M(N) \rangle = N$.

3. CLASSICAL PRIMARY SUBSEMIMODULES

In this section, we give some characterizations for classical primary subsemimodules of R -semimodule M .

Lemma 3.1. Let $M = M_1 \times M_2$, where M_i is an R_i -semimodule. A subsemimodule $N_1 \times M_2$ is a classical primary subsemimodule of M if and only if N_1 is a classical primary subsemimodule of M_1 .

Proof. Suppose that $N_1 \times M_2$ is a classical primary subsemimodule of R -semimodule M . We will show that N_1 is a classical primary subsemimodule of M_1 . Clearly, N_1 is a proper subsemimodule of R_1 -semimodule M_1 . To show that classical primary subsemimodule properties of N_1 hold, $m \in M_1$ and $a, b \in R_1$ such that $abm \in N_1$. Then

$$(a, 1)(b, 1)(m, n) = (abm, n) \in N_1 \times M_2.$$

Since $N_1 \times M_2$ is a classical primary subsemimodule of R -semimodule M , it follows that

$$(am, n) = (a, 1)(m, n) \in N_1 \times M_2$$

or

$$(b^n m, n) = (b, 1)^n(m, n) \in N_1 \times M_2,$$

for some positive integer n . That is, $am \in N_1$ or $b^n m \in N_1$. Therefore N_1 is a classical primary subsemimodule of R_1 -semimodule M_1 . Conversely, suppose that N_1 is a classical primary subsemimodule of R_1 -semimodule M_1 . We will show that $N_1 \times M_2$ is a classical primary subsemimodule of R -semimodule M . Clearly, $N_1 \times M_2$ is a proper subsemimodule of R -semimodule M . To show that classical primary subsemimodule properties of $N_1 \times M_2$ hold, let $(m, n) \in M$ and $(a_1, a_2), (b_1, b_2) \in R$ such that

$$(a_1 b_1 m, a_2 b_2 n) = (a_1, a_2)(b_1, b_2)(m, n) \in N_1 \times M_2.$$

Since N_1 is a classical primary subsemimodule of R_1 -semimodule M_1 and $a_1 b_1 m \in N_1$, we have $a_1 m \in N_1$ or $b_1^n m \in N_1$, for some positive integer n . Therefore

$$(a_1, a_2)(m, n) = (a_1 m, a_2 n) \in N_1 \times M_2$$

or

$$(b_1, b_2)^n(m, n) = (b_1^n m, b_2^n n) \in N_1 \times M_2.$$

Hence $N_1 \times M_2$ is a classical primary subsemimodule of R -semimodule M .

4. RADICAL OF CLASSICAL PRIMARY SUBSEMIMODULES

A subsemimodule N of an R -semimodule M is said to satisfy the classical primary radical formula in M , if $\langle E_M(N) \rangle = c.prad_M(N)$.

Lemma 4.1. Let $M = M_1 \times M_2$, where M_i is an R_i -semimodule. If W is a classical primary subsemimodule of R -semimodule M and $P = \{x \in M_1 : (x, y) \in W\}$, then $P = M_1$ or P is a classical primary subsemimodule of R_1 -semimodule M_1 .

Proof. Suppose that $P \neq M_1$. We will show that P is a classical primary subsemimodule of R_1 -semimodule M_1 . It is clear that, P is a proper subsemimodule of R_1 -semimodule M_1 . To show that classical primary subsemimodule properties of P , let $a, b \in R_1$ and $m \in M_1$ such that $abm \in P$. Then $(a, 1)(b, 1)(m, y) = (abm, y) \in W$. Since W is a classical primary subsemimodule of M , we have

$$(am, 1) = (a, 1)(m, y) \in W$$

or

$$(b^n m, y) = (b, 1)^n (m, y) \in W,$$

for some positive integer n . It follows that $am \in P$ or $b^n m \in P$. Therefore P is a classical primary subsemimodule of M_1 .

Corollary 4.2. Let $M = M_1 \times M_2$, where M_i is an R_i -semimodule. If W is a classical primary subsemimodule of R -semimodule M and $P = \{x \in M_2 : (0, x) \in W\}$, then $P = M_2$ or P is a classical primary subsemimodule of R_2 -semimodule M_2 .

Proof. This follows from Lemma 4.1.

Corollary 4.3. Let $M = \prod_{i=1}^n M_i$, where M_i is an R_i -semimodule. If W is a classical primary subsemimodule of R -semimodule M and $P = \{x \in M_j : (m_1, m_2, \dots, x, m_{j+1}, \dots, m_n) \in W\}$, then $P = M_j$ or P is a classical primary subsemimodule of R_j -semimodule M_j .

Proof. This follows from Lemma 4.1 and Corollary 4.2.

Lemma 4.4. Let $M = M_1 \times M_2$, where M_i is an R_i -semimodule and let N be a subsemimodule of R_1 -semimodule M_1 . Then $m \in c.prad_{M_1}(N)$ if and only if $(m, y) \in c.prad_M(N \times \{y\})$.

Proof. Suppose that $M = M_1 \times M_2$, where M_i is an R_i -semimodule. Let N be a subsemimodule of R_1 -semimodule M_1 and let $m \in c.prad_{M_1}(N)$.

If there is no classical primary subsemimodule of M containing $N \times \{y\}$, then $c.prad_M(N \times \{y\}) = M$. Therefore $(m, y) \in c.prad_M(N \times \{y\})$.

Corollary 4.8. Let $M = \prod_{i=1}^n M_i$, where M_i is an R_i -semimodule. If N_i be a subsemimodule of R_i -semimodule M_i ,

then $\prod_{i=1}^n c.prad_{M_i}(N_i) \subseteq c.prad_M(\prod_{i=1}^n N_i)$.

Proof. This follows from Lemma 4.7.

Theorem 4.9. Let $M = M_1 \times M_2$, where M_i is an R_i -semimodule. If N is a subsemimodule of R_1 -semimodule M_1 , then $c.prad_{M_1}(N) \times c.prad_{M_2}(M_2) = c.prad_M(N \times M_2)$.

Proof. Suppose that $M = M_1 \times M_2$, where M_i is an R_i -semimodule. Let N be a subsemimodule of R_1 -semimodule M_1 . By Lemma 4.7, we have $c.prad_{M_1}(N) \times c.prad_{M_2}(M_2) \subseteq c.prad_M(N \times M_2)$. We will show that $c.prad_M(N \times M_2) \subseteq c.prad_{M_1}(N) \times c.prad_{M_2}(M_2)$. If there is no classical primary subsemimodule of M containing N , then $c.prad_{M_1}(N) = M_1$. Then

$$c.prad_M(N \times M_2) \subseteq c.prad_{M_1}(N) \times c.prad_{M_2}(M_2).$$

If there is classical primary subsemimodule of M containing N , then there exists W is a classical primary subsemimodule of M_1 containing N . Then $W \times M_2$ is a classical primary subsemimodule of M , containing $N \times M_2$. Let P be a classical primary subsemimodule of M containing $N \times M_2$. Then

$$\begin{aligned} N \times M_2 &\subseteq c.prad_{M_1}(N) \times M_2 \\ &= c.prad_{M_1}(N) \times c.prad_{M_2}(M_2). \end{aligned}$$

Therefore $c.prad_M(N \times M_2) \subseteq c.prad_{M_1}(N) \times c.prad_{M_2}(M_2)$ and hence

$$c.prad_M(N \times M_2) = c.prad_{M_1}(N) \times c.prad_{M_2}(M_2).$$

Corollary 4.10. Let $M = M_1 \times M_2$, where M_i is an R_i -semimodule. If N is a subsemimodule of R_2 -semimodule M_2 , then $c.prad_M(M_1 \times N) = c.prad_{M_1}(M_1) \times c.prad_{M_2}(N)$.

Proof. This follows from Lemma 4.9.

Corollary 4.11. Let $M = \prod_{i=1}^n M_i$, where M_i is an R_i -semimodule. If N_i be a subsemimodule of R_i -semimodule

M_i , then $\prod_{i=1}^n c.prad_{M_i}(N_i) = c.prad_M(\prod_{i=1}^n N_i)$.

Proof. This follows from Lemma 4.9 and Corollary 4.10.

Theorem 4.12. Let $M = M_1 \times M_2$, where M_i is an R_i -semimodule. If N_1 is a classical primary subsemimodule of M_1 , then N_1 is to satisfy the classical primary radical formula in M_1 if and only if $N_1 \times M_2$ is to satisfy the classical primary radical formula in M .

Proof. Suppose that N_1 is a classical primary subsemimodule of M_1 and N_1 is to satisfy the classical primary radical formula in M_1 . We will show that $N_1 \times M_2$ is to satisfy the classical primary radical formula in M . Since N_1 is a classical primary subsemimodule of M_1 , it follows that

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