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# Existence theorems of the hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term in Banach spaces

Rabian Wangkeeree · Pakkapon Preechasilp

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**Abstract** In this paper, we establish some existence results for the hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term in reflexive Banach spaces. Using the concept of the stable  $f$ -quasimonotonicity, the properties of Clarke's generalized directional derivative, Clarke's generalized gradient and KKM technique, some existence theorems of solutions are proved when the constrained set is nonempty, bounded (or unbounded), closed and convex. Our main results extend various results existing in the current literatures.

**Keywords** Hemivariational inequality governed by a multi-valued map · Clarke's generalized gradient · KKM technique · Banach space

## 1 Introduction

The study of inequality problems captured special attention in the last decades, one of the most recent and general type of inequalities being the hemivariational inequalities. The notion of hemivariational inequality was introduced by Panagiotopoulos [33,37] at the beginning of the 1980s as a variational formulation for several classes of mechanical problems with nonsmooth and nonconvex energy super-potentials. In the case of convex super-potentials, hemivariational inequalities reduce to variational inequalities which were studied earlier by many authors (see e.g. Fichera [14] or Hartman and Stampacchia [20]). For example, Carl [4], Carl et al. [5,6] and Xiao and Huang [46] studied the existence of solutions of some kinds of hemi-variational inequalities using the method of sub-super solutions. Migorski and Ochal [28], and Park and Ha [39,40] studied the problem using the regularized approximating method. Goeleven et al. [17] and Liu [27] proved the existence of solutions using the method

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of the first eigenfunction. For more related works regarding the existence of solutions for hemivariational inequalities, we refer to [8–10, 15, 16, 24, 28, 31–34, 36–39, 44] and references therein. Recently, Zhang and He [49] established some existence results for hemivariational inequalities of the Hartman-Stampacchia type involving stably quasimonotone set-valued mappings on bounded, closed and convex subsets in reflexive Banach spaces. Very recently, Tang and Huang [45] studied the existence of solutions for the variational-hemivariational inequalities in reflexive Banach spaces. By using the concept of stable  $\phi$ -quasimonotonicity and the properties of Clarke's generalized directional derivative and Clarke's generalized gradient, they obtained some existence theorems when the constrained set is nonempty, bounded (or unbounded), closed and convex.

On the other hand, the minimax inequality was first discovered by Ky Fan in 1972 reads as follows (see [12]). Let  $K$  be a nonempty compact convex subset of a Hausdorff topological vector space  $E$ ,  $f : K \times K \rightarrow \mathbb{R}$ . Assume that  $f$  fulfills the following properties:

- (i) for each fixed  $y \in K$ ,  $f(\cdot, y)$  is lower semicontinuous;
- (ii) for each fixed  $x \in K$ ,  $f(x, \cdot)$  is quasi-concave;
- (iii) for all  $x \in K$ ,  $f(x, x) \leq 0$ .

Then, there exists  $y_0 \in K$  such that

$$f(y_0, x) \leq 0, \quad \text{for all } x \in K. \quad (1)$$

The famous Ky Fan's minimax inequality (1) is one of the most important tools in nonlinear analysis and in mathematical economics. Indeed, it allows one to derive several practical and theoretical results in many fields of mathematics, including optimization-related problems. The Ky Fan's minimax inequality was shown to be equivalent to powerful theorems in nonlinear analysis, from Brouwer's fixed point theorem, the fixed point theorem of Kakutani, the fixed point theorem of Ky Fan-Glicksberg, the corpus of analysis to many other fixed-point theorems, the lemma of Knaster-Kuratowski-Mazurkiewicz, the theorem of Browder concerning variational inequalities, the Nash equilibrium principle, the theorem of Gale-Nikaido-Debreu on Walras equilibrium in mathematical economics, etc. (see [47, vol. IV, pp. 795–810]). Since that time, the Ky Fan's minimax inequality have been improving in many directions; see also [3, 24, 41, 44]. It is clear that the Ky Fan's inequalities and the hemivariational inequalities are two different and useful generalizations of the variational inequality in several aspects. It is natural to study these different problems in a unified framework. This fact motivated us to consider the class of generalized Ky Fan-hemiequilibrium problems as follows.

Let  $K$  be a nonempty, closed and convex subset of a real reflexive Banach space  $E$  with its dual  $E^*$ ,  $F : K \rightrightarrows 2^{E^*}$  a multivalued mapping. Let  $\Omega$  be a bounded open set in  $\mathbb{R}^N$ ,  $T : E \rightarrow L^p(\Omega; \mathbb{R}^k)$  a linear continuous mapping, where  $1 < p < \infty$ ,  $k \geq 1$  and  $j : \Omega \times \mathbb{R}^k \rightarrow \mathbb{R}$  a function. We shall denote  $\hat{u} := Tu$ ,  $j^\circ(x, y; h)$  denotes the Clarke's generalized directional derivative of a locally Lipschitz mapping  $j(x, \cdot)$  at the point  $y \in \mathbb{R}^k$  with respect to direction  $h \in \mathbb{R}^k$ , where  $x \in \Omega$ . For the bifunction  $f : K \times K \rightarrow [-\infty, +\infty]$  imposed the condition that the set  $\mathcal{D}_1(f) = \{u \in K : f(u, v) \neq -\infty, \forall v \in K\}$  is nonempty, in this paper, we discuss the following *hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term* :

$$(HVI MN) \begin{cases} \text{Find } u \in \mathcal{D}_1(f) \text{ and } u^* \in F(u) \text{ such that} \\ \langle u^*, v - u \rangle + f(u, v) + \int_{\Omega} j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x)) dx \geq 0, \\ \forall v \in K. \end{cases} \quad (2)$$

Now, let us consider some special cases of the problem (2).

- (a) If  $f(u, v) = \phi(v) - \phi(u)$ , where  $\phi : X \rightarrow \mathbb{R} \cup \{+\infty\}$  is a proper, convex and lower semicontinuous function such that  $K_\phi = K \cap \text{dom}\phi \neq \emptyset$ , then  $\mathcal{D}_1(f) = K_\phi$ , (2) is reduced to the following *variational-hemivariational inequality problem*: Find  $u \in K_\phi$  such that

$$\langle u^*, v - u \rangle + \phi(v) - \phi(u) + \int_{\Omega} j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x)) dx \geq 0, \quad \forall v \in K. \quad (3)$$

This inequality was studied by Costea and Lupu [8] by assuming that  $F$  is monotone and lower hemicontinuous and several existence results were obtained. Recently, these results were extended by Tang and Huang [45] by assuming that  $F$  is stably  $\phi$ -quasimonotone with respect to a certain subset  $U$  of  $X^*$ . In the case when  $F$  is single valued, monotone and hemicontinuous (3) was first studied by Motreanu and Radulescu [30].

- (b) If  $F \equiv 0$  and  $f(u, v) = \Lambda(u, v) - \langle g^*, v - u \rangle$ , where  $\Lambda : K \times K \rightarrow \mathbb{R}$  and  $g^* \in X^*$ , then (2) reduces to the problem: Find  $u \in K$  such that

$$\Lambda(u, v) + \int_{\Omega} j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x)) dx \geq \langle g^*, v - u \rangle, \quad \forall v \in K. \quad (4)$$

This inequality was studied by Costea and Radulescu [9] and it was called nonlinear hemivariational inequality (see also Andrei and Costea [1] for some applications of nonlinear hemivariational inequalities to Nonsmooth Mechanics).

- (c) If  $f(u, v) \equiv 0$ , then (2) becomes the problem : Find  $u \in K$  and  $u^* \in F(u)$  such that

$$\langle u^*, v - u \rangle + \int_{\Omega} j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x)) dx \geq 0, \quad \forall v \in K. \quad (5)$$

This inequality was recently studied by Zhang and He [49] by assuming that  $F$  is stably quasimonotone with respect to some subset  $U$  of  $X^*$  and lower hemicontinuous and by assuming the  $F$  is upper semicontinuous with compact convex values (see Zhang and He [48]). When  $F$  is single-valued inequality (5) was studied by Panagiotopoulos et al. [38] by assuming the  $F$  is weak\*-demicontinuous or monotone and hemicontinuous and by Costea and Radulescu [9] by assuming that  $F$  is stably pseudomonotone and hemicontinuous.

- (d) Costea and Radulescu [11] studied very recently the following quasi-hemivariational inequality problem : Find  $u \in K$  and  $u^* \in F(u)$  such that

$$\langle u^*, v - u \rangle + H(u) \int_{\Omega} j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x)) dx \geq \langle G(u), v - u \rangle, \quad \forall v \in K, \quad (6)$$

where  $G : X \rightarrow X^*$  is a nonlinear single-valued operator. Taking  $f(u, v) = -\langle G(u), v - u \rangle$ , inequality (2) can be regarded as a generalization of (6) in the case  $H \equiv 1$ . Also, when  $F$  is single-valued and  $H \equiv 1$  inequality (6) was investigated by Repovš and Varga [43] and then applied to obtain some existence results for systems of hemivariational results.

- (e) If  $F$  is single-valued and  $f(u, v) = \int_{\Omega} g(x, \hat{u}(x))(\hat{v}(x) - \hat{u}(x)) dx$ , then (2) reduces to the problem : Find  $u \in K$  such that

$$\begin{aligned} & \langle F(u), v - u \rangle + \int_{\Omega} g(x, \hat{u}(x))(\hat{v}(x) - \hat{u}(x))dx \\ & + \int_{\Omega} j^{\circ}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0, \quad \forall v \in K, \end{aligned} \quad (7)$$

which was studied by Lisei et al. [25] in the case when  $\Omega \subset \mathbb{R}^N$  is open and unbounded.

- (f) If  $F \equiv 0$ ,  $T \equiv 0$  and  $f(u, v) = \varphi(u, v) - \langle g^*, v - u \rangle$ , with  $g^* \in X^*$  given, then (2) reduces to the following nonlinear inequality problem : Find  $u \in K$  such that

$$\varphi(u, v) \geq \langle g^*, v - u \rangle, \quad \forall v \in K. \quad (8)$$

Inequality (8) was studied by Ky Fan [12] when  $g^* = 0$ , and by Gwinner [18] (in reflexive Banach spaces) and Ansari et al. [2] (in locally convex spaces) when  $g^* \neq 0$ .

The aim of this paper is to study the existence of solutions for *HVIMN* (2) in a reflexive Banach space. To establish our results, we derive the notion of stable  $f$ -quasimonotonicity and use the properties of Clarke's generalized directional derivative, Clarke's generalized gradient, and KKM technique. Some existence results of solutions are proved when the constrained set  $K$  is bounded, closed and convex set. In case  $K$  is an unbounded set, the coercive conditions are also derived to utilize for our results. The results presented in this paper extend and improve some known results.

This paper is divided into 4 sections. After we review literature, we next recall some definitions and useful tools in Sect. 2. In Sect. 3, we introduce some kinds of generalized monotonicity of the operator and discuss the relations of these generalized monotonicity in details. In Sect. 4, we prove the existence of solutions when  $K$  is bounded and unbounded set in Theorems 4.1 and 4.5, respectively. In Theorem 4.7, we give a necessary and sufficient condition to the boundedness of the solution set of *HVIMN*.

## 2 Preliminaries

Let  $E$  be a reflexive Banach space with norm denoted by  $\|\cdot\|$  and  $E^*$  be its dual space. Let  $K$  be a nonempty, closed and convex subset of  $E$ . For every  $r > 0$ , we define

$$B_r = \{y \in K : \|x\| < r\}.$$

We recall that a function  $\varphi : E \rightarrow \mathbb{R}$  is called locally Lipschitz if for every  $x \in E$  there exists a neighborhood  $U$  of  $x$  and a constant  $L_x \geq 0$ , so called Lipschitz constant, such that

$$|\varphi(v) - \varphi(w)| \leq L_x \|v - w\|, \quad \forall v, w \in U.$$

Let  $T : E \rightarrow L^p(\Omega; \mathbb{R}^k)$  be a linear compact operator, where  $1 < p < \infty$  and  $k \geq 1$ , and  $\Omega$  a bounded open subset of  $\mathbb{R}^N$ . We shall denote  $\hat{u} = T(u)$  and  $q$  by the conjugated exponent of  $p$ , i.e.,  $\frac{1}{p} + \frac{1}{q} = 1$ . Let  $j : \Omega \times \mathbb{R}^k \rightarrow \mathbb{R}$  be a function such that the mapping

$$j(\cdot, y) : \Omega \rightarrow \mathbb{R} \text{ is measurable, } \quad \forall y \in \mathbb{R}^k. \quad (j1)$$

We assume that the following conditions holds: either there exists  $l \in L^q(\Omega; \mathbb{R})$  such that

$$|j(x, y_1) - j(x, y_2)| \leq l(x)|y_1 - y_2|, \quad \forall x \in \Omega, \quad \forall y_1, y_2 \in \mathbb{R}^k, \quad (j2)$$

or

$$\text{the mapping } j(x, \cdot) \text{ is locally Lipschitz, } \quad \forall x \in \Omega, \quad (j3)$$

and there exists  $C > 0$  such that

$$|z| \leq C(1 + |y|^{p-1}), \quad \forall x \in \Omega, \forall z \in \partial j(x, y). \quad (\text{j4})$$

Recall that  $g^\circ(x; v)$  denotes Clarke's generalized directional derivative of the locally Lipschitz mapping  $g : E \rightarrow \mathbb{R}$  at the point  $x \in E$  with respect to the direction  $v \in E$ , while  $\partial g(x)$  is the Clarke's generalized gradient of  $g$  at  $x \in E$  (see [7]), i.e.,

$$g^\circ(x; v) = \limsup_{y \rightarrow x, t \downarrow 0} \frac{g(y + tv) - g(y)}{t}$$

and

$$\partial g(x) = \{\xi \in E^* : \langle \xi, v \rangle \leq g^\circ(x; v), \forall v \in E\},$$

where  $\langle \cdot, \cdot \rangle$  denotes the duality pairing between  $E^*$  and  $E$ . Let  $J : L^p(\Omega; \mathbb{R}^k) \rightarrow \mathbb{R}$  be an arbitrary locally Lipschitz functional. For each  $u \in E$  there exists (see, for example [7])  $z_u \in \partial J(\hat{u})$  such that

$$J^\circ(\hat{u}; \xi) = \langle z_u, \xi \rangle = \max \{\langle w, \xi \rangle, w \in \partial J(\hat{u})\}.$$

Denoting by  $T^* : L^q(\Omega; \mathbb{R}^k) \rightarrow E^*$  the adjoint operator of  $T$ , i.e.,  $\langle x, T^*y^* \rangle = \langle Tx, y^* \rangle$  for all  $x \in E$  and  $y^* \in L^q(\Omega; \mathbb{R}^k)$ . We define the subset  $U(J, T)$  of  $E^*$  as follows:

$$U(J, T) = \{-z_u^* : u \in K, z_u^* = T^*z_u\}. \quad (9)$$

**Remark 2.1** It is easy to obtain that  $J^\circ(\hat{u}; \hat{v} - \hat{u}) = \langle z_u^*, v - u \rangle$  for all  $u, v \in E$ . Indeed, Since  $T^*$  is an adjoint operator of  $T$ , for any  $u, v \in E$ , we have

$$\begin{aligned} J^\circ(\hat{u}; \hat{v} - \hat{u}) &= \langle z_u, \hat{v} - \hat{u} \rangle \\ &= \langle z_u, T(v) - T(u) \rangle \\ &= \langle z_u, T(v - u) \rangle \\ &= \langle T^*z_u, v - u \rangle \\ &= \langle z_u^*, v - u \rangle. \end{aligned}$$

The following results are useful to our proof.

**Lemma 2.2** [7, Proposition 2.1.1] *Let  $g : K \rightarrow \mathbb{R}$  be Lipschitz of rank  $L_x$  near  $x$ . Then*

1. *the function  $v \mapsto g^\circ(x; v)$  is finite, positively homogeneous, and subadditive on  $E$ , and satisfies*

$$|g^\circ(x; v)| \leq L_x \|v\|;$$

2.  *$g^\circ(x; v)$  is upper semicontinuous as a function of  $(x, v)$  and, as a function of  $v$  alone, is Lipschitz of rank  $L_x$  on  $E$ ;*
3.  *$g^\circ(x; -v) = (-g)^\circ(x; v)$ .*

**Lemma 2.3** [7, Proposition 2.7.5] *If  $J(\varphi) = \int_\Omega j(x, \varphi(x))dx$ , and  $j$  satisfies either (j1) and (j2) or (j1) and (j3)-(j4), then  $J$  is uniformly Lipschitz on bounded subsets, and one has*

$$\partial J(\varphi) \subset \int_\Omega \partial j(x, \varphi(x))dx.$$

*Further, if  $j$  is regular at  $(x, \varphi(x))$  then  $J$  is regular at  $\varphi$  at equality holds.*

The following version of the KKM theorem is due to Ky Fan [13].

**Lemma 2.4** [13] *Let  $K$  be a nonempty subset of a Hausdorff topological vector space  $X$  and let  $G : K \rightrightarrows X$  be a multivalued mapping satisfying the following properties:*

1.  *$G$  is a KKM mapping, i.e.,*

$$\text{co}\{x_1, x_2, \dots, x_n\} \subset \bigcup_{i=1}^n G(x_i), \text{ for } x_i \in K, i = 1, 2, \dots, n;$$

2.  *$G(x)$  is closed in  $X$  for every  $x \in K$ ;*
3.  *$G(x_0)$  is compact in  $X$  for some  $x_0 \in K$ .*

*Then  $\bigcap_{x \in K} G(x) \neq \emptyset$ .*

The following definition can be founded in [23].

**Definition 2.5** Let  $F : K \rightarrow E^*$  be a multivalued mapping. the  $F$  is said to be

1. lower semicontinuous at  $x_0$  if, for any  $x_0^* \in F(x_0)$  and sequence  $\{x_n\} \subset K$  with  $x_n \rightarrow x_0$ , there exists a sequence  $x_n^* \in F(x_n)$  which converges to  $x_0^*$ ;
2. lower hemicontinuous if, the restriction of  $F$  to every line segment of  $K$  is lower semicontinuous with respect to the weak topology in  $E^*$ .

The following definitions (i), (ii) and (iii) are well-known, (iv) and (v) can be found in [49], and (vi)–(ix) can be found in [45].

**Definition 2.6** Let  $\phi : K \rightarrow \mathbb{R}$  be a function, and  $F : K \rightrightarrows E^*$  a multivalued mapping. Then  $F$  is said to be

1. monotone if, for each  $u, v \in K$ ,

$$\langle v^* - u^*, v - u \rangle \geq 0, \quad \forall u^* \in F(u) \text{ and } v^* \in F(v);$$

2. pseudomonotone if, for each  $u, v \in K$ ,

$$\langle u^*, v - u \rangle \geq 0 \implies \langle v^*, v - u \rangle \geq 0, \quad \forall u^* \in F(u) \text{ and } v^* \in F(v);$$

3. quasimonotone if, for each  $u, v \in K$ ,

$$\langle u^*, v - u \rangle > 0 \implies \langle v^*, v - u \rangle \geq 0, \quad \forall u^* \in F(u) \text{ and } v^* \in F(v);$$

4. stably pseudomonotone with respect to the set  $U \subset E^*$  if,  $F$  and  $F(\cdot) - \xi$  are pseudomonotone for every  $\xi \in U$ ;

5. stably quasimonotone with respect to the set  $U \subset E^*$  if,  $F$  and  $F(\cdot) - \xi$  are quasimonotone for every  $\xi \in U$ ;

6.  $\phi$ -pseudomonotone if, for each  $u, v \in K$ ,

$$\begin{aligned} \langle u^*, v - u \rangle + \phi(v) - \phi(u) &\geq 0 \implies \langle v^*, v - u \rangle + \phi(v) - \phi(u) \\ &\geq 0, \quad \forall u^* \in F(u) \text{ and } v^* \in F(v); \end{aligned}$$

7.  $\phi$ -quasimonotone if, for each  $u, v \in K$ ,

$$\begin{aligned} \langle u^*, v - u \rangle + \phi(v) - \phi(u) &> 0 \implies \langle v^*, v - u \rangle + \phi(v) - \phi(u) \\ &\geq 0, \quad \forall u^* \in F(u) \text{ and } v^* \in F(v); \end{aligned}$$

8. stably  $\phi$ -pseudomonotone with respect to the set  $U \subset E^*$  if,  $F$  and  $F(\cdot) - \xi$  are  $\phi$ -pseudomonotone for every  $\xi \in U$ ;

9. stably  $\phi$ -quasimonotone with respect to the set  $U \subset E^*$  if,  $F$  and  $F(\cdot) - \xi$  are  $\phi$ -quasimonotone for every  $\xi \in U$ .

### 3 Generalized monotonicity

First, we give the concept of stable  $f$ -quasimonotonicity with respect to the set  $U \subset E^*$  which is useful for establishing existence theorems for the main results.

**Definition 3.1** Let  $f : K \times K \rightarrow \mathbb{R}$  be a bifunction, and  $F : K \rightrightarrows E^*$  a multivalued mapping. Then  $F$  is said to be

1.  $f$ -pseudomonotone if, for each  $u, v \in K$ ,  

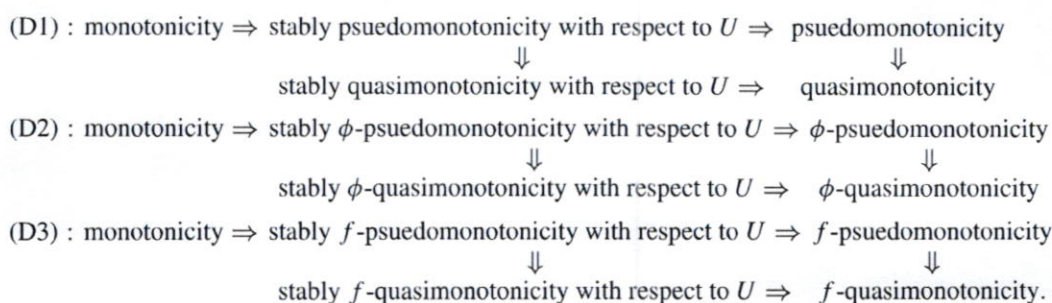
$$\langle u^*, v - u \rangle + f(u, v) \geq 0 \implies \langle v^*, v - u \rangle + f(u, v) \geq 0, \quad \forall u^* \in F(u) \text{ and } v^* \in F(v);$$
2.  $f$ -quasimonotone if, for each  $u, v \in K$ ,  

$$\langle u^*, v - u \rangle + f(u, v) > 0 \implies \langle v^*, v - u \rangle + f(u, v) \geq 0, \quad \forall u^* \in F(u) \text{ and } v^* \in F(v);$$
3. stably  $f$ -pseudomonotone with respect to the set  $U \subset E^*$  if,  $F$  and  $F(\cdot) - \xi$  are  $f$ -pseudomonotone for every  $\xi \in U$ ;
4. stably  $f$ -quasimonotone with respect to the set  $U \subset E^*$  if,  $F$  and  $F(\cdot) - \xi$  are  $f$ -quasimonotone for every  $\xi \in U$ .

**Remark 3.2** 1. If  $f(u, v) = \phi(v) - \phi(u)$ , where  $\phi : K \rightarrow \mathbb{R}$ , then the notations of  $f$ -pseudomonotonicity,  $f$ -quasimonotonicity, stable  $f$ -pseudomonotonicity and stable  $f$ -quasimonotonicity of the operator  $F$  reduce to ones of  $\phi$ -pseudomonotonicity,  $\phi$ -quasimonotonicity, stable  $\phi$ -pseudomonotonicity and stable  $\phi$ -quasimonotonicity of the operator  $F$ , respectively. Those concepts are introduced in [45].

2. If  $f \equiv 0$ , then the notations of  $f$ -pseudomonotonicity,  $f$ -quasimonotonicity, stable  $f$ -pseudomonotonicity and stable  $f$ -quasimonotonicity of the operator  $F$  reduce to ones of pseudomonotonicity, quasimonotonicity, stable pseudomonotonicity and stable quasimonotonicity of the operator  $F$ , respectively. Those concepts are introduced in [49].

**Remark 3.3** We represent the implications between monotonicity and some kinds of generalized monotonicity through the following diagrams:



The inverse direction of each implication relationship mentioned by three diagrams above does not hold in general. Example 2 in [21] had shown that a stably pseudomonotone mapping with respect to a closed line segment is not necessarily monotone; Example 3.1 in [19] had shown that a quasimonotone mapping may not be pseudomonotone; Example 4.1 in [49] had shown that a stably quasimonotone mapping is not necessarily stable pseudomonotone; Example 3.1 in [45] had shown that stably  $\phi$ -pseudomonotone mapping may not be monotone, while Example 4.1 in the same paper had shown that a stably  $\phi$ -quasimonotone mapping is not necessarily stably  $\phi$ -pseudomonotone and  $\phi$ -quasimonotone mapping may not be  $\phi$ -pseudomonotone.

Let  $A, A_\phi$  and  $A_f$  denote the sets consisting of generalized monotonicity in diagrams (D1), (D2) and (D3), respectively. Example 3.1 [45] had shown that each of generalized

monotonicity taken from the set  $A$  is independent of any one taken from  $A_\phi$  if  $\phi = I_K$ . Also, the following two examples illustrate that each of generalized monotonicity taken from the set  $A_\phi$  is independent of any one taken from  $A_f$  if  $f(u, v) \neq \phi(v) - \phi(u)$ , where  $\phi : K \rightarrow \mathbb{R}$ .

**Example 3.4** Let  $E = \mathbb{R}^2$  and  $K = [3, 5] \times \{0\}$ . Let  $f : K \times K \rightarrow [-\infty, +\infty]$ ,  $F : K \rightarrow E^* = \mathbb{R}^2$  and  $\phi : K \rightarrow \mathbb{R}$  by

$$f(x, y) = y_1^2 - x_1 y_1 \text{ and } F(x) = [-5, 1] \times \{0\} \text{ and } \phi(x) = x_1^2,$$

where  $x = (x_1, 0)$ ,  $y = (y_1, 0)$ . Then  $F$  is stably  $\phi$ -pseudomonotone with respect to the set  $V := \{(0, m) : m \in \mathbb{R}\}$  but not  $f$ -quasimonotone. Indeed, we first show that  $F$  is  $\phi$ -pseudomonotone on  $K$ . If

$$\begin{aligned} 0 \leq \langle x^*, y - x \rangle + \phi(y) - \phi(x) &= x_1^*(y_1 - x_1) + y_1^2 - x_1^2 \\ &= (x_1^* + y_1 + x_1)(y_1 - x_1). \end{aligned}$$

Since  $x_1^* \in [-5, 1]$  and  $x_1, y_1 \in [3, 5]$ , we get that  $x_1^* + y_1 + x_1 > 0$ . It implies that  $y_1 - x_1 \geq 0$ . Thus, we have

$$\langle y^*, y - x \rangle + \phi(y) - \phi(x) = (y_1^* + y_1 + x_1)(y_1 - x_1) \geq 0.$$

Hence  $F$  is  $\phi$ -pseudomonotone on  $K$ .

Next, we show that  $F(\cdot) - \xi$  is  $\phi$ -pseudomonotone for each  $\xi = (0, m) \in V$ . If

$$\begin{aligned} 0 \leq \langle x^* - \xi, y - x \rangle + \phi(y) - \phi(x) &= x_1^*(y_1 - x_1) + y_1^2 - x_1^2 \\ &= (x_1^* + y_1 + x_1)(y_1 - x_1). \end{aligned}$$

Since  $x_1^* \in [-5, 1]$  and  $x_1, y_1 \in [3, 5]$ , we get that  $x_1^* + y_1 + x_1 > 0$ . It implies that  $y_1 - x_1 \geq 0$ . Thus, we have

$$\langle y^* - \xi, y - x \rangle + \phi(y) - \phi(x) = (y_1^* + y_1 + x_1)(y_1 - x_1) \geq 0.$$

Hence  $F$  is stably  $\phi$ -pseudomonotone with respect to the set  $V$ .

Finally, we show that  $F$  is not  $f$ -quasimonotone on  $K$ . Take  $x = (3, 0)$ ,  $y = (4, 0)$ ,  $x^* = (1, 0)$  and  $y^* = (-5, 0)$ . Then we have

$$\begin{aligned} \langle x^*, y - x \rangle + f(x, y) &= x_1^*(y_1 - x_1) + y_1(y_1 - x_1) \\ &= (x_1^* + y_1)(y_1 - x_1) \\ &= (1 + 4)(4 - 3) > 0, \end{aligned}$$

but

$$\begin{aligned} \langle y^*, y - x \rangle + f(x, y) &= y_1^*(y_1 - x_1) + y_1(y_1 - x_1) \\ &= (y_1^* + y_1)(y_1 - x_1) \\ &= (-5 + 4)(4 - 3) < 0, \end{aligned}$$

Hence  $F$  is not  $f$ -quasimonotone on  $K$ .  $\square$

**Example 3.5** Let  $E = \mathbb{R}^2$  and  $K = [-1, 2] \times \{0\}$ . Let  $f : K \times K \rightarrow [-\infty, +\infty]$ ,  $F : K \rightarrow E^* = \mathbb{R}^2$  and  $\phi : K \rightarrow \mathbb{R}$  by

$$f(x, y) = y_1^2 - x_1 y_1 \text{ and } F(x) = \left[ \frac{4}{3}, 3 \right] \times \{0\} \text{ and } \phi(x) = x_1^2,$$

where  $x = (x_1, 0)$ ,  $y = (y_1, 0)$ . Then  $F$  is  $f$ -pseudomonotone with respect to the set  $V := \{(0, m) : m \in \mathbb{R}\}$  but not  $\phi$ -quasimonotone. Indeed, we first show that  $F$  is  $f$ -pseudomonotone on  $K$ . If

$$\begin{aligned} 0 \leq \langle x^*, y - x \rangle + f(x, y) &= x_1^*(y_1 - x_1) + y_1(y_1 - x_1) \\ &= (x_1^* + y_1)(y_1 - x_1). \end{aligned}$$

Since  $x_1^* + y_1 > 0$ , we have  $y_1 - x_1 \geq 0$ .

Thus

$$\begin{aligned} 0 \leq \langle y^*, y - x \rangle + f(x, y) &= y_1^*(y_1 - x_1) + y_1(y_1 - x_1) \\ &= (y_1^* + y_1)(y_1 - x_1). \end{aligned}$$

Hence  $F$  is  $f$ -pseudomonotone on  $K$ .

Next, we show that  $F(\cdot) - \xi$  is pseudomonotone for all  $\xi \in V$ . If

$$\begin{aligned} 0 \leq \langle x^* - \xi, y - x \rangle + f(x, y) &= x_1^*(y_1 - x_1) + y_1(y_1 - x_1) \\ &= (x_1^* + y_1)(y_1 - x_1). \end{aligned}$$

Since  $x_1^* + y_1 > 0$ , we have  $y_1 - x_1 \geq 0$ . Thus

$$\begin{aligned} 0 \leq \langle y^*, y - x \rangle + f(x, y) &= y_1^*(y_1 - x_1) + y_1(y_1 - x_1) \\ &= (y_1^* + y_1)(y_1 - x_1). \end{aligned}$$

Hence  $F$  is stably  $f$ -pseudomonotone with respect to the set  $V$ .

Finally, we show that  $F$  is not  $\phi$ -quasimonotone on  $K$ . If we take  $x = (-1, 0)$ ,  $y = (-\frac{1}{2}, 0)$ ,  $x^* = (3, 0)$  and  $y^* = (\frac{4}{3}, 0)$ , then we have that

$$\begin{aligned} \langle x^*, y - x \rangle + \phi(y) - \phi(x) &= x_1^*(y_1 - x_1) + y_1^2 - x_1^2 \\ &= x_1^*(y_1 - x_1) + y_1^2 - x_1^2 - x_1 y_1 + x_1 y_1 \\ &= (x_1^* + x_1 + y_1)(y_1 - x_1) \\ &= \left(3 - 1 - \frac{1}{2}\right) \left(-\frac{1}{2} - (-1)\right) = \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4} > 0, \end{aligned}$$

but

$$\begin{aligned} \langle y^*, y - x \rangle + \phi(y) - \phi(x) &= y_1^*(y_1 - x_1) - y_1^2 + x_1^2 \\ &= y_1^*(y_1 - x_1) - y_1^2 + x_1^2 - x_1 y_1 + x_1 y_1 \\ &= (y_1^* - x_1 - y_1)(y_1 - x_1) < 0 \\ &= \left(\frac{4}{3} - 1 - \frac{1}{2}\right) \left(-\frac{1}{2} - (-1)\right) = -\frac{1}{6} \cdot \frac{1}{2} = -\frac{1}{12} < 0. \end{aligned}$$

Hence  $F$  is not  $\phi$ -quasimonotone on  $K$ .  $\square$

#### 4 Existence of solutions and coerciveness conditions

In this section, we present the existence of solutions for  $HVIMN$  when  $K$  is a bounded subset of  $E$ .

A bifunction  $f : K \times K \rightarrow [-\infty, +\infty]$  is said to satisfy the condition  $\Delta$  if

- (i)  $\mathcal{D}_1(f) = \{u \in K : f(u, v) \neq -\infty, \forall v \in K\}$  is nonempty;
- (ii)  $f(x, x) = 0$  for all  $x \in K$ ;
- (iii)  $f(x, y) + f(y, x) \geq 0$  for all  $x, y \in K$ ;
- (iv) for all  $y \in K$ ,  $f(\cdot, y)$  is weakly upper semicontinuous;
- (v) for all  $x \in K$ ,  $f(x, \cdot)$  is convex.

**Theorem 4.1** Let  $K$  be a nonempty, closed, bounded and convex subset of a real reflexive Banach space  $E$ . Let  $f : K \times K \rightarrow [-\infty, +\infty]$  be a bifunction satisfying the condition  $\Delta$ . Let  $J : L^p(\Omega; \mathbb{R}^k) \rightarrow \mathbb{R}$  be a function

$$J(\varphi) = \int_{\Omega} j(x, \varphi(x)) dx,$$

and  $T : E \rightarrow L^p(\Omega; \mathbb{R}^k)$  be a linear compact operator, where  $1 < p < \infty$ ,  $k \geq 1$  and  $\Omega$  is a bounded open set in  $\mathbb{R}^N$ . Let  $F : K \rightrightarrows E^*$  be a lower hemicontinuous set-valued mapping and stably  $f$ -quasimonotone with respect to the set  $U(J, T)$  defined in (9). Suppose further that  $j$  satisfies either (j1) and (j2) or (j1) and (j3)–(j4). Then HVIMN has a solution.

*Proof* For any  $v \in K$  define a multivalued mapping  $G : K \rightrightarrows E$  as follows:

$$G(v) := \left\{ u \in K : \inf_{v^* \in F(v)} \langle v^*, v - u \rangle + f(u, v) + J^\circ(\hat{u}; \hat{v} - \hat{u}) \geq 0 \right\}. \quad (10)$$

Consider two cases of  $G$ : (a)  $G$  is not a KKM mapping, and (b)  $G$  is a KKM mapping.

**Case (a)** If  $G$  is not a KKM mapping, then there exist  $u_i \in K$  and  $\lambda_i \in [0, 1]$ ,  $i = 1, 2, \dots, N$ , with  $\sum_{i=1}^N \lambda_i = 1$  such that  $u_0 = \sum_{i=1}^N \lambda_i u_i \notin \bigcup_{i=1}^N G(u_i)$ , that is,

$$\inf_{u_i^* \in F(u_i)} \langle u_i^*, u_i - u_0 \rangle + f(u_0, u_i) + J^\circ(\hat{u}_0; \hat{u}_i - \hat{u}_0) < 0, \forall i \in \{1, 2, \dots, N\}. \quad (11)$$

We claim that there exists a neighborhood  $U$  of  $u_0$  such that for all  $v \in U \cap K$ ,

$$\inf_{u_i^* \in F(u_i)} \langle u_i^*, u_i - v \rangle + f(v, u_i) + J^\circ(\hat{v}; \hat{u}_i - \hat{v}) < 0, \forall i \in \{1, 2, \dots, N\}.$$

If not, for any neighborhood  $U$  of  $u_0$ , there exist  $v_0 \in U \cap K$  and  $i_0 \in \{1, 2, \dots, N\}$  such that

$$\inf_{u_{i_0}^* \in F(u_{i_0})} \langle u_{i_0}^*, u_{i_0} - v_0 \rangle + f(v_0, u_{i_0}) + J^\circ(\hat{v}_0; \hat{u}_{i_0} - \hat{v}_0) \geq 0.$$

Putting  $U = B(u_0, \frac{1}{n})$ , so there exists  $v_n \in B(u_0, \frac{1}{n}) \cap K$  such that

$$\inf_{u_{i_0}^* \in F(u_{i_0})} \langle u_{i_0}^*, u_{i_0} - v_n \rangle + f(v_n, u_{i_0}) + J^\circ(\hat{v}_n; \hat{u}_{i_0} - \hat{v}_n) \geq 0.$$

By Lemma 2.2 (ii),  $v_n \rightarrow u_0$  and  $\Delta$  (iv), we obtain that

$$\inf_{u_{i_0}^* \in F(u_{i_0})} \langle u_{i_0}^*, u_{i_0} - u_0 \rangle + f(u_0, u_{i_0}) + J^\circ(\hat{v}_n; \hat{u}_{i_0} - \hat{u}_0) \geq 0,$$

which is a contradiction with (11), so we have the claim.

From Remark 2.1 and (11), there exists a neighborhood  $U$  of  $u_0$  such that for all  $v \in U \cap K$ ,

$$\begin{aligned} & \inf_{u_i^* \in F(u_i)} \langle u_i^*, u_i - v \rangle + f(v, u_i) + \langle z_v^*, u_i - v \rangle \\ &= \inf_{u_i^* \in F(u_i)} \langle u_i^*, u_i - v \rangle + f(v, u_i) + J^\circ(\hat{v}; \hat{u}_i - \hat{v}) \\ &< 0, \forall i \in \{1, 2, \dots, N\}, \end{aligned}$$

which can be rewritten as

$$\inf_{u_i^* \in F(u_i)} \langle u_i^* - (-z_v^*), u_i - v \rangle + f(v, u_i) < 0, \forall i \in \{1, 2, \dots, N\}.$$

By the stable  $f$ -quasimonotonicity of  $F$  with respect to the set  $U(J, T)$ , we get that

$$\sup_{v^* \in F(v)} \langle v^* - (-z_v^*), u_i - v \rangle + f(v, u_i) \leq 0, \quad \forall i \in \{1, 2, \dots, N\},$$

which is equivalent to

$$\sup_{v^* \in F(v)} \langle v^*, u_i - v \rangle + f(v, u_i) + J^\circ(\hat{v}; \hat{u}_i - \hat{v}) \leq 0, \quad \forall i \in \{1, 2, \dots, N\}.$$

From (i) of Lemma 2.2,  $\Delta(v)$  and the linearity of  $T$ , we have that

$$\begin{aligned} & \sup_{v^* \in F(v)} \langle v^*, u_0 - v \rangle + f(v, u_0) + J^\circ(\hat{v}; \hat{u}_0 - \hat{v}) \\ &= \sup_{v^* \in F(v)} \left\langle v^*, \sum_{i=1}^N \lambda_i u_i - v \right\rangle + f\left(v, \sum_{i=1}^N \lambda_i u_i\right) + J^\circ\left(\hat{v}; \sum_{i=1}^N \lambda_i \hat{u}_i - \hat{v}\right) \\ &\leq \sum_{i=1}^N \lambda_i \left[ \sup_{v^* \in F(v)} \langle v^*, u_i - v \rangle + f(v, u_i) + J^\circ(\hat{v}; \hat{u}_i - \hat{v}) \right] \\ &\leq 0. \end{aligned} \quad (12)$$

By subadditivity of  $v \mapsto J^\circ(x; v)$ , gives that

$$J^\circ(\hat{v}; \hat{v} - \hat{u}_0) + J^\circ(\hat{v}; \hat{u}_0 - \hat{v}) \geq J^\circ(\hat{v}; 0) = 0.$$

Combining the last inequality with (12), we obtain that

$$\inf_{v^* \in F(v)} \langle v^*, v - u_0 \rangle - f(v, u_0) + J^\circ(\hat{v}; \hat{v} - \hat{u}_0) \geq 0, \quad \forall v \in U \cap K. \quad (13)$$

It follows from  $\Delta$  (iii) that

$$\inf_{v^* \in F(v)} \langle v^*, v - u_0 \rangle + f(u_0, v) + J^\circ(\hat{v}; \hat{v} - \hat{u}_0) \geq 0, \quad \forall v \in U \cap K. \quad (14)$$

Let  $v' \in K$  be any element and define

$$u_m = \frac{1}{m} v' + \left(1 - \frac{1}{m}\right) u_0 \text{ for all } m \in \mathbb{N}.$$

Thus,  $u_m \rightarrow u_0$  as  $m \rightarrow \infty$  and hence there exists  $M \in \mathbb{N}$  such that

$$u_m \in U \cap K \text{ for all } m > M.$$

For any given  $u_0^* \in F(u_0)$ , since  $F$  is lower hemicontinuous, there exists a sequence  $\{u_m^*\}$  in  $F(u_m)$  converging weakly star to  $u_0^*$ . It follows from (14) that for any  $m > M$ ,

$$\langle u_m^*, u_m - u_0 \rangle + f(u_0, u_m) + J^\circ(\hat{u}_m; \hat{u}_m - \hat{u}_0) \geq 0.$$

By (i) of Lemma 2.2, linearity of  $T$ , (A1) and (A4), we have that

$$\begin{aligned} 0 &\leq \left\langle u_m^*, \frac{1}{m}(v' - u_0) \right\rangle + f\left(u_0, u_0 + \frac{1}{m}(v' - u_0)\right) + J^\circ\left(\hat{u}_m; \frac{1}{m}(\hat{v}' - \hat{u}_0)\right) \\ &\leq \frac{1}{m} \left[ \langle u_m^*, v' - u_0 \rangle + f(u_0, v') + J^\circ(\hat{u}_m; \hat{v}' - \hat{u}_0) \right]. \end{aligned} \quad (15)$$

Multiplying the last inequality by  $m$  and letting  $m \rightarrow \infty$ , from (ii) of Lemma 2.2, we obtain that

$$\langle u_0^*, v' - u_0 \rangle + f(u_0, v') + J^\circ(\hat{u}_0; \hat{v}' - \hat{u}_0) \geq 0, \quad \forall v' \in K. \quad (16)$$

Since  $J(\varphi) = \int_{\Omega} j(x, \varphi(x))dx$  and  $j$  satisfies either (j1) and (j2) or (j1) and (j3)–(j4), by Lemma 2.3, we get that

$$\int_{\Omega} j^{\circ}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq J^{\circ}(\hat{u}; \hat{v} - \hat{u}), \quad \forall u, v \in K.$$

It follows from (16) that

$$\langle u_0^*, v' - u_0 \rangle + f(u_0, v') + \int_{\Omega} j^{\circ}(x, \hat{u}_0(x); \hat{v}'(x) - \hat{u}_0(x))dx \geq 0, \quad \forall v' \in K.$$

Hence  $HVIMN$  has a solution.

**Case (b)** If  $G$  is a KKM mapping. We consider the following mapping :

$$u \mapsto \inf_{v^* \in F(v)} \langle v^*, v - u \rangle + f(u, v) + J^{\circ}(\hat{u}; \hat{v} - \hat{u}). \quad (17)$$

We claim that the above mapping is weakly upper semicontinuous. For given a sequence  $\{\mu_n\} \subset K$  such that  $\mu_n \rightharpoonup \mu_0$ , it follows from linearity and compactness of  $T$  that  $T\mu_n \rightarrow T\mu_0$ ; that is  $\hat{\mu}_n \rightarrow \hat{\mu}_0$  as  $n \rightarrow \infty$ . Since  $f(\cdot, y)$  is weakly upper semicontinuous for all  $y \in K$  and Lemma 2.2 (ii), we have that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left[ \inf_{v^* \in F(v)} \langle v^*, v - \mu_n \rangle + f(\mu_n, v) + J^{\circ}(\hat{\mu}_n; \hat{v} - \hat{\mu}_n) \right] \\ & \leq \limsup_{n \rightarrow \infty} \left( \inf_{v^* \in F(v)} \langle v^*, v - \mu_n \rangle \right) + \limsup_{n \rightarrow \infty} f(\mu_n, v) + \limsup_{n \rightarrow \infty} J^{\circ}(\hat{\mu}_n; \hat{v} - \hat{\mu}_n) \\ & \leq \inf_{v^* \in F(v)} \langle v^*, v - \mu_0 \rangle + f(\mu_0, v) + J^{\circ}(\hat{\mu}_0; \hat{v} - \hat{\mu}_0). \end{aligned}$$

Thus we have the claim. Then  $G(v)$  is weakly closed. It follows from the convexity boundedness and closedness of a subset  $K$  in a reflexive Banach space  $E$ , we have that  $K$  is weakly compact. Since  $G(v) \subset K$ , we get that  $G(v)$  is weakly compact for each  $v \in K$ . Thus, all conditions of Lemma 2.4 are satisfied in the weak topology and so we obtain that  $\bigcap_{v \in K} G(v) \neq \emptyset$ . Taking  $u_0 \in \bigcap_{v \in K} G(v)$ , we have

$$\inf_{v^* \in F(v)} \langle v^*, v - u_0 \rangle + f(u_0, v) + J^{\circ}(\hat{u}_0; \hat{v} - \hat{u}_0) \geq 0, \quad \forall v \in K. \quad (18)$$

Let  $v_0 \in K$  be arbitrarily fixed and define  $u_n = u_0 + \frac{1}{n}(v_0 - u_0)$ . Then  $u_n \in K$  for all  $n \geq 1$ . For any  $u_0^* \in F(u_0)$ , since  $F$  is lower hemicontinuous, there exists a sequence  $\{u_n^*\}$  in  $F(u_n)$  converging weakly star to  $u_0^*$ . From (18), for any  $n \geq 1$ , we have

$$\langle u_n^*, u_n - u_0 \rangle + f(u_0, u_n) + J^{\circ}(\hat{u}_0; \hat{u}_n - \hat{u}_0) \geq 0.$$

In view of the linearity of  $T$ ,  $\Delta(v)$  and Lemma 2.2 (i), we have

$$\begin{aligned} 0 & \leq \langle u_n^*, \frac{1}{n}(v_0 - u_0) \rangle + f\left(u_0, u_0 + \frac{1}{n}(v_0 - u_0)\right) + J^{\circ}\left(\hat{u}_0; \frac{1}{n}(\hat{v}_0 - \hat{u}_0)\right) \\ & \leq \frac{1}{n} [\langle u_n^*, v_0 - u_0 \rangle + f(u_0, v_0) + J^{\circ}(\hat{u}_0; \hat{v}_0 - \hat{u}_0)]. \end{aligned}$$

Multiplying the inequality above by  $n$  and letting  $n \rightarrow \infty$ , we obtain that

$$\langle u_0^*, v_0 - u_0 \rangle + f(u_0, v_0) + J^{\circ}(\hat{u}_0; \hat{u}_0 - \hat{u}_0) \geq 0, \quad \forall v_0 \in K. \quad (19)$$

Since  $J(\varphi) = \int_{\Omega} j(x, \varphi(x))dx$  and  $j$  satisfies either (j1) and (j2) or (j1) and (j3)–(j4), by Lemma 2.3, we get that

$$\int_{\Omega} j^{\circ}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x)) dx \geq J^{\circ}(\hat{u}; \hat{v} - \hat{u}), \quad \forall u, v \in K.$$

This together with (19), we get

$$\langle u_0^*, v_0 - u_0 \rangle + f(u_0, v_0) + \int_{\Omega} j^{\circ}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x)) dx \geq 0, \quad \forall v_0 \in K.$$

Hence  $u_0$  is a solution of  $HVIMN$ . This completes the proof.  $\square$

**Remark 4.2** Theorem 4.1 generalizes and improves some recent results. In fact,

1. If  $f(u, v) = \phi(v) - \phi(u)$ , where  $\phi : X \rightarrow \mathbb{R} \cup \{+\infty\}$  is a proper, convex and lower semicontinuous function such that  $K_{\phi} = K \cap \text{dom}\phi \neq \emptyset$ , then  $\mathcal{D}_1(f) = K_{\phi}$ , then Theorem 4.1 reduces to Theorem 4.1 of Tang and Huang [45];
2. If  $f \equiv 0$ , then Theorem 4.1 reduces to Theorem 3.1 of Zhang and He [49];
3. If  $f \equiv 0$  and the mapping  $F$  in Theorem 4.1 is single-valued, then Theorem 4.1 extends Corollary 1 of Costea and Rădulescu [9] by relaxing the stably pseudomonotonicity of  $F$  in Corollary 1 of [9] to  $F$  being stably quasimonotone;
4. Theorem 4.1 generalizes and improves Theorem 2 of Motreanu and Rădulescu [30] by extending  $F$  from the single-valued case to the set-valued one and by relaxing the monotonicity of  $F$  in Theorem 2 of [30] to  $F$  being stably  $f$ -quasimonotone;
5. If  $f \equiv 0$  and the mapping  $F$  is single-valued, then Theorem 4.1 generalizes and improves Theorem 2 of Panagiotopolos et al. [38] by relaxing the monotonicity of  $F$  in [38] to  $F$  being stably quasimonotone and by extending  $F$  from the single-valued case to the set-valued one.

Next, we omit the boundedness of  $K$  in Theorem 4.1, we need to introduce the concept of  $f$ -coercivity.

**Proposition 4.3** Consider the following  $f$ -coercivity conditions:

- (A) There exists a nonempty subset  $V_0$  contained in a weakly compact subset  $V_1$  of  $K$  such that the set

$$D = \{u \in K : \inf_{v^* \in F(v)} \langle v^*, v - u \rangle + f(u, v) + J^{\circ}(\hat{u}; \hat{v} - \hat{u}) \geq 0, \quad \forall v \in V_0\}$$

is weakly compact or empty.

- (B) There exists  $n_0 \in \mathbb{N}$  such that for every  $u \in K \setminus B_{n_0}$ , there exists some  $v \in K$  with  $\|v\| < \|u\|$  such that

$$\sup_{u^* \in F(u)} \langle u^*, v - u \rangle + f(u, v) + J^{\circ}(\hat{u}; \hat{v} - \hat{u}) \leq 0.$$

- (C) There exists  $n_0 \in \mathbb{N}$  such that for every  $u \in K \setminus B_{n_0}$ , there exists some  $v \in K$  with  $\|v\| < \|u\|$  such that

$$\sup_{u^* \in F(u)} \langle u^*, v - u \rangle + f(u, v) + \int_{\Omega} j^{\circ}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x)) dx < 0.$$

Then we have

1. (A)  $\Rightarrow$  (B), if  $F$  is stably  $f$ -quasimonotone with respect to the set  $U(J, T)$ .
2. (C)  $\Rightarrow$  (B), if  $J(\varphi) = \int_{\Omega} j(x, \varphi(x)) dx$ ,  $j$  satisfies either (j1) and (j2) or (j1) and (j3)–(j4).

*Proof* (i) If  $D = \emptyset$ , since  $V_0$  is nonempty and contained in a weakly compact subset  $V_1$  of  $K$ , then there exists a natural number  $M < \infty$  such that  $\|z\| < M$  for all  $z \in V_0$ . Taking  $n_0 = M$ , we obtain that for every  $u \in K \setminus B_{n_0}$ , there exists  $v \in V_0 \neq \emptyset$  such that  $v \in B_{n_0}$  and

$$\inf_{v^* \in F(v)} \langle v^*, v - u \rangle + f(u, v) + J^\circ(\hat{u}; \hat{v} - \hat{u}) < 0. \quad (20)$$

If  $D \neq \emptyset$ , then  $D$  is weakly compact. Since  $D \cup V_0 \subset D \cup V_1$ , which is a weakly compact subset, we conclude that there exists a natural number  $M < \infty$  such that  $\|z\| < M$  for all  $z \in D \cup V_0$ . Taking  $n_0 = M$ , for every  $u \in K \setminus B_{n_0}$  and (20) holds. Hence, from the proves of both case we can conclude that there exists  $n_0 \in \mathbb{N}$  such that, for any  $u \in K \setminus B_{n_0}$ , there exists  $v \in B_{n_0}$  such that (20) holds. Therefore,

$$\begin{aligned} 0 &> \inf_{v^* \in F(v)} \langle v^*, v - u \rangle + f(u, v) + \langle z_u^*, v - u \rangle \\ &= \inf_{v^* \in F(v)} \langle v^*, v - u \rangle + f(u, v) + J^\circ(\hat{u}; \hat{v} - \hat{u}), \end{aligned}$$

and hence

$$\sup_{v^* \in F(v)} \langle v^* - (-z_u^*), u - v \rangle + f(u, v) > 0.$$

Since  $F$  is stably  $f$ -quasimonotonicity with respect to  $U(J, T)$ , we have

$$\inf_{u^* \in F(u)} \langle u^* - (-z_u^*), u - v \rangle + f(u, v) \geq 0,$$

and then

$$\sup_{u^* \in F(u)} \langle u^*, v - u \rangle + f(u, v) + J^\circ(\hat{u}; \hat{v} - \hat{u}) \leq 0.$$

This complete the proof.  $\square$

(ii) By Lemma 2.3, we get that

$$\int_{\Omega} j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x)) dx \geq J^\circ(\hat{u}; \hat{v} - \hat{u}), \quad \forall u, v \in E.$$

Combining with (C), we get (B).

- Remark 4.4** 1. If  $f(u, v) = \phi(v) - \phi(u)$ , where  $\phi : K \rightarrow \mathbb{R}$ , then the  $f$ -coercivity conditions of Proposition 4.3 reduce to  $\phi$ -coercivity conditions of Proposition 4.1 in [45].
2. If  $f(u, v) \equiv 0$ , then the  $f$ -coercivity conditions of Proposition 4.3 reduce to coercivity conditions of Proposition 3.1 in [49].

Next, we present the main theorem for the unbounded constrained set.

**Theorem 4.5** *Let  $K$  be a nonempty, closed, unbounded and convex subset of a real reflexive Banach space  $E$ . Let  $f : K \times K \rightarrow [-\infty, +\infty]$  be a bifunction satisfying the condition  $\Delta$ . Let  $J : L^p(\Omega; \mathbb{R}^k) \rightarrow \mathbb{R}$  be a function*

$$J(\varphi) = \int_{\Omega} j(x, \varphi(x)) dx,$$

and  $T : E \rightarrow L^p(\Omega; \mathbb{R}^k)$  be a linear compact operator, where  $1 < p < \infty$ ,  $k \geq 1$  and  $\Omega$  is a bounded open set in  $\mathbb{R}^N$ . Let  $F : K \rightrightarrows E^*$  be a lower hemicontinuous set-valued mapping and stably  $f$ -quasimonotone with respect to the set  $U(J, T)$  defined in (9). Suppose further that  $j$  satisfies either (j1) and (j2) or (j1) and (j3)–(j4). If the condition (B) holds, then  $HVIMN$  has a solution.

*Proof* Take  $m > n_0$ . Since  $B_m$  is bounded and convex, from (16) or (19) in Theorem 4.1, we can conclude that there exists  $u_m \in B_m \cap K$  and  $u_m^* \in F(u_m)$  such that

$$\langle u_m^*, v - u_m \rangle + f(u_m, v) + J^\circ(\hat{u}_m; \hat{v} - \hat{u}_m) \geq 0, \quad \forall v \in B_m \cap K. \quad (21)$$

We consider two cases.

- (a) If  $\|u_m\| = m$ , then  $\|u_m\| > n_0$ . Since the condition (B) holds, there is some  $v_0 \in K$  with  $\|v_0\| < \|u_m\|$  such that

$$\langle u_m^*, v_0 - u_m \rangle + f(u_m, v_0) + J^\circ(\hat{u}_m; \hat{v}_0 - \hat{u}_m) \leq 0. \quad (22)$$

Let  $v \in K$ . Since  $\|v_0\| < \|u_m\| = m$ , there is  $t \in (0, 1)$  such that  $v_t = v_0 + t(v - v_0) \in B_m \cap K$ . Notice that  $T$  is a linear mapping and  $f(x, \cdot)$  is a convex function. By (21), (22) and (i) of Lemma 2.2, It follows that

$$\begin{aligned} 0 &\leq \langle u_m^*, v_t - u_m \rangle + f(u_m, v_t) + J^\circ(\hat{u}_m; \hat{v}_t - \hat{u}_m) \\ &= \langle u_m^*, (1-t)v_0 + tv - u_m \rangle + f(u_m, (1-t)v_0 + tv) \\ &\quad + J^\circ(\hat{u}_m; (1-t)\hat{v}_0 + t\hat{v} - \hat{u}_m) \\ &\leq (1-t) [\langle u_m^*, v_0 - u_m \rangle + f(u_m, v_0) + J^\circ(\hat{u}_m; \hat{v}_0 - \hat{u}_m)] \\ &\quad + t [\langle u_m^*, v - u_m \rangle + f(u_m, v) + J^\circ(\hat{u}_m; \hat{v} - \hat{u}_m)] \\ &\leq t [\langle u_m^*, v - u_m \rangle + f(u_m, v) + J^\circ(\hat{u}_m; \hat{v} - \hat{u}_m)], \quad \forall v \in K. \end{aligned} \quad (23)$$

Dividing by  $t$ , we have that

$$\langle u_m^*, v - u_m \rangle + f(u_m, v) + J^\circ(\hat{u}_m; \hat{v} - \hat{u}_m) \geq 0, \quad \forall v \in K. \quad (24)$$

- (b) If  $\|u_m\| < m$ , then for any  $v \in K$ , there is  $t \in (0, 1)$  such that  $v_t = n_m + t(v - v_m) \in B_m \cap K$ . Notice that  $T$  is a linear mapping and  $f(x, \cdot)$  is a convex function. By (21) and (i) of Lemma 2.2, It follows that

$$\begin{aligned} 0 &\leq \langle u_m^*, v_t - u_m \rangle + f(u_m, v_t) + J^\circ(\hat{u}_m; \hat{v}_t - \hat{u}_m) \\ &\leq t [\langle u_m^*, v - u_m \rangle + f(u_m, v) + J^\circ(\hat{u}_m; \hat{v} - \hat{u}_m)], \quad \forall v \in K. \end{aligned}$$

Dividing by  $t$ , we have that (24) holds.

Since  $J(\varphi) = \int_{\Omega} j(x, \varphi(x))dx$  and  $j$  satisfies either (j1) and (j2) or (j1) and (j3)–(j4), by Lemma 2.3, we have that

$$\int_{\Omega} j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq J^\circ(\hat{u}; \hat{v} - \hat{u}), \quad \forall u, v \in E,$$

and so

$$\begin{aligned} 0 &\leq \langle u_m^*, v - u_m \rangle + f(u_m, v) + J^\circ(\hat{u}_m; \hat{v} - \hat{u}_m) \\ &\leq \langle u_m^*, v - u_m \rangle + f(u_m, v) + \int_{\Omega} j^\circ(x, \hat{u}_m(x); \hat{v}(x) - \hat{u}_m(x))dx, \quad \forall v \in K. \end{aligned}$$

This shows that  $HVIMN$  has a solution.  $\square$

- Remark 4.6** 1. If  $f(u, v) = \phi(v) - \phi(u)$ , where  $\phi : K \rightarrow \mathbb{R}$ , then Theorem 4.5 reduces to Theorem 4.2 of [45].
2. If  $f \equiv 0$ , then Theorem 4.5 reduces to Theorem 3.2 of [49]. Theorem 4.5 also generalizes and improves Theorem 2 of [9] by extending  $F$  from single-valued case to set-valued one and relaxing the corresponding coercivity condition and stable pseudomonotonicity of the operator in [9].

If the constraint set  $K$  is bounded, then the solution set of  $HVIMN$  is obviously bounded. In the case of constraint set  $K$  is unbounded, the solution set of  $HVIMN$  may be unbounded. In the sequel, we provide a sufficient condition to the boundedness of the solution set of  $HVIMN$ , when  $K$  is unbounded. The following theorem also generalizes Theorem 4.3 of [45].

**Theorem 4.7** *Let  $K$  be a nonempty, closed, unbounded and convex subset of a real reflexive Banach space  $E$ . Let  $f : K \times K \rightarrow [-\infty, +\infty]$  be a bifunction satisfying the condition  $\Delta$ . Let  $J : L^p(\Omega; \mathbb{R}^k) \rightarrow \mathbb{R}$  be a function*

$$J(\varphi) = \int_{\Omega} j(x, \varphi(x)) dx,$$

*and  $T : E \rightarrow L^p(\Omega; \mathbb{R}^k)$  be a linear compact operator, where  $1 < p < \infty, k \geq 1$  and  $\Omega$  is a bounded open set in  $\mathbb{R}^N$ . Let  $F : K \rightrightarrows E^*$  be a lower hemicontinuous set-valued mapping and stably  $f$ -quasimonotone with respect to the set  $U(J, T)$  defined in (9). Suppose further that  $j$  satisfies either (j1) and (j2) or (j1) and (j3)–(j4). If the condition (C) holds, then  $VHIEP$  is nonempty and bounded.*

*Proof* From Proposition 4.3, we have (C)  $\Rightarrow$  (B). By Theorem 4.5, we know that the solution set of the problem (2) is nonempty. If the solution set is unbounded, then there exist  $u_0 \in K$  and  $u_0^* \in F(u_0)$  such that  $\|u_0\| > n_0$  and

$$\langle u_0^*, v - u_0 \rangle + f(u_0, v) + \int_{\Omega} j^\circ(x, \hat{u}_0(x); \hat{v}(x) - \hat{u}_0(x)) dx \geq 0, \quad \forall v \in K. \quad (25)$$

Since  $\|u_0\| > n_0$ , it follows from condition (C) that, there exists  $v_0 \in K$  with  $\|v_0\| < \|u_0\|$  such that

$$\sup_{u^* \in F(u_0)} \langle u^*, v_0 - u_0 \rangle + f(u_0, v_0) + \int_{\Omega} j^\circ(x, \hat{u}_0(x); \hat{v}_0(x) - \hat{u}_0(x)) dx < 0,$$

which is a contradiction with (25). This complete the proof.  $\square$

- Remark 4.8** 1. If  $f(u, v) = \phi(v) - \phi(u)$ , where  $\phi : K \rightarrow \mathbb{R}$ , then Theorem 4.7 reduces to Theorem 4.3 of [45].
2. If  $f \equiv 0$ , then Theorem 4.7 reduces to Theorem 3.3 of [49].

## 5 Conclusions

In this paper, we establish some existence results for the Ky Fan-hemivariational inequality problems when the set  $K$  is bounded in a reflexive Banach space by using the notion of stable  $f$ -quasimonotonicity and use the properties of Clarke's generalized directional derivative, Clarke's generalized gradient, and KKM technique. When the set  $K$  is unbounded, we give a coerciveness condition for the existence of the solution and a coerciveness condition for

the existence and boundedness of solution. The results presented in this paper extend and improve some known results.

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