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Viscosity approximation methods for nonexpansive semigroups in CAT(0) spaces

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Abstract

In this paper, we study the strong convergence of Moudafi's viscosity approximation methods for approximating a common fixed point of a one-parameter continuous semigroup of nonexpansive mappings in CAT(0) spaces. We prove that the proposed iterative scheme converges strongly to a common fixed point of a one-parameter continuous semigroup of nonexpansive mappings which is also a unique solution of the variational inequality. The results presented in this paper extend and enrich the existing literature.

Keywords: viscosity approximation method; nonexpansive semigroup; variational inequality; CAT(0) space; common fixed point

1 Introduction

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic* (or metric) *segment* joining x and y . When it is unique, this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for all $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

CAT(0): Let Δ be a geodesic triangle in X and let $\bar{\Delta}$ be a comparison triangle for Δ . Then Δ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y}).$$

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \quad (1.1)$$

This is the (CN)-inequality of Bruhat and Tits [1]. In fact (cf. [2], p.163), a geodesic space is a CAT(0) space if and only if it satisfies the (CN)-inequality.

It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include pre-Hilbert spaces, $\mathbb{R}$ -trees (see [2]), Euclidean buildings (see [3]), the complex Hilbert ball with a hyperbolic metric (see [4]), and many others. Complete CAT(0) spaces are often called Hadamard spaces.

It is proved in [2] that a normed linear space satisfies the (CN)-inequality if and only if it satisfies the parallelogram identity, i.e., is a pre-Hilbert space; hence it is not so unusual to have an inner product-like notion in Hadamard spaces. Berg and Nikolaev [5] introduced the concept of quasilinearization as follows.

Let us formally denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then *quasilinearization* is defined as a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \rightarrow \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2} (d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)) \quad (a, b, c, d \in X). \quad (1.2)$$

It is easily seen that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a, b, c, d, x \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d) \quad (1.3)$$

for all $a, b, c, d \in X$. It is known [5, Corollary 3] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.

In 2010, Kakavandi and Amini [6] introduced the concept of a dual space for CAT(0) spaces as follows. Consider the map $\Theta : \mathbb{R} \times X \times X \rightarrow C(X)$ defined by

$$\Theta(t, a, b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \quad (1.4)$$

where $C(X)$ is the space of all continuous real-valued functions on X . Then the Cauchy-Schwarz inequality implies that $\Theta(t, a, b)$ is a Lipschitz function with a Lipschitz semi-norm $L(\Theta(t, a, b)) = |t|d(a, b)$ for all $t \in \mathbb{R}$ and $a, b \in X$, where

$$L(f) = \sup \left\{ \frac{f(x) - f(y)}{d(x, y)} : x, y \in X, x \neq y \right\}$$

is the Lipschitz semi-norm of the function $f : X \rightarrow \mathbb{R}$. Now, define the pseudometric D on $\mathbb{R} \times X \times X$ by

$$D((t, a, b), (s, c, d)) = L(\Theta(t, a, b) - \Theta(s, c, d)).$$

Lemma 1.1 [6, Lemma 2.1] $D((t, a, b), (s, c, d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$ for all $x, y \in X$.

For a complete CAT(0) space (X, d) , the pseudometric space $(\mathbb{R} \times X \times X, D)$ can be considered as a subspace of the pseudometric space $(\text{Lip}(X, \mathbb{R}), L)$ of all real-valued Lipschitz functions. Also, D defines an equivalence relation on $\mathbb{R} \times X \times X$, where the equivalence class of $\vec{tab} := (t, a, b)$ is

$$[\vec{tab}] = \{ \vec{sca} : t(\vec{ab}, \vec{xy}) = s(\vec{cd}, \vec{xy}) \forall x, y \in X \}.$$

The set $X^* := \{ [\vec{tab}] : (t, a, b) \in \mathbb{R} \times X \times X \}$ is a metric space with metric D , which is called the dual metric space of (X, d) .

Recently, Dehghan and Roojin [7] introduced the duality mapping in CAT(0) spaces and studied its relation with subdifferential, by using the concept of quasilinearization. Then they presented a characterization of metric projection in CAT(0) spaces as follows.

Theorem 1.2 [7, Theorem 2.4] *Let C be a nonempty convex subset of a complete CAT(0) space X , $x \in X$ and $u \in C$. Then*

$$u = P_C x \quad \text{if and only if} \quad \langle \vec{yu}, \vec{ux} \rangle \geq 0 \quad \text{for all } y \in C.$$

From now on, let $\mathbb{N}$ be the set of positive integers, let $\mathbb{R}$ be the set of real numbers, and let $\mathbb{R}^+$ be the set of nonnegative real numbers. Let C be a nonempty, closed and convex subset of a complete CAT(0) space X . A family $\mathcal{S} := \{T(t) : t \in \mathbb{R}^+\}$ of self-mappings of C is called a one-parameter continuous semigroup of nonexpansive mappings if the following conditions hold:

- (i) for each $t \in \mathbb{R}^+$, $T(t)$ is a nonexpansive mapping on C , i.e.,

$$d(T(t)x, T(t)y) \leq d(x, y), \quad \forall x, y \in C;$$

- (ii) $T(s+t) = T(t) \circ T(s)$ for all $t, s \in \mathbb{R}^+$;

- (iii) for each $x \in X$, the mapping $T(\cdot)x$ from $\mathbb{R}^+$ into C is continuous.

A family $\mathcal{S} := \{T(t) : t \in \mathbb{R}^+\}$ of mappings is called a one-parameter strongly continuous semigroup of nonexpansive mappings if conditions (i), (ii) and (iii) and the following condition are satisfied:

- (iv) $T(0)x = x$ for all $x \in C$.

We shall denote by $\mathcal{F}$ the common fixed point set of $\mathcal{S}$, that is,

$$\mathcal{F} := F(\mathcal{S}) = \{x \in C : T(t)x = x, t \in \mathbb{R}^+\} = \bigcap_{t \in \mathbb{R}^+} F(T(t)).$$

One classical way to study nonexpansive mappings is to use contractions to approximate nonexpansive mappings. More precisely, take $t \in (0, 1)$ and define a contraction $T_t : C \rightarrow C$ by

$$T_t = tu + (1-t)Tx, \quad \forall x \in C,$$

where $u \in C$ is an arbitrary fixed element. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in C . It is unclear, in general, what the behavior of x_t is as $t \rightarrow 0$, even if T has a fixed point. However, in the case of T having a fixed point,

Browder [8] proved that x_t converges strongly to a fixed point of T that is nearest to u in the framework of Hilbert spaces. Reich [9] extended Browder's result to the setting of Banach spaces and proved, in a uniformly smooth Banach space, that x_t converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from C onto $F(T)$.

Halpern [10] introduced the following explicit iterative scheme (1.5) for a nonexpansive mapping T on a subset C of a Hilbert space by taking any points $u, x_1 \in C$ and defined the iterative sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n. \quad (1.5)$$

He proved that the sequence $\{x_n\}$ generated by (1.5) converges to a fixed point of T .

It is an interesting problem to extend the above (Browder's [8] and Halpern's [10]) results to the nonexpansive semigroup case. In [11], Shioji and Takahashi introduced the following implicit iteration in a Hilbert space:

$$x_n = \alpha_n u + (1 - \alpha_n) \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds, \quad (1.6)$$

where C is a nonempty closed convex subset of a real Hilbert space H , $u \in C$, $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{t_n\}$ is a sequence of positive real numbers divergent to ∞ . Under suitable conditions, they proved strong convergence of $\{x_n\}$ to a member of F .

Later, Suzuki [12] was the first to introduce in a Hilbert space the following iteration process:

$$x_n = \alpha_n u + (1 - \alpha_n)T(t_n)x_n, \quad \forall n \geq 1, \quad (1.7)$$

where $\{T(t) : t \geq 0\}$ is a strongly continuous semigroup of nonexpansive mappings on C such that $F \neq \emptyset$ and $\{\alpha_n\}$ and $\{t_n\}$ are appropriate sequences of real numbers. He proved that $\{x_n\}$ generated by (1.7) converges strongly to the element of F nearest to u . Using Moudafi's viscosity approximation methods, Song and Xu [13] introduced the following iteration process:

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n, \quad \forall n \geq 1, \quad (1.8)$$

and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n, \quad \forall n \geq 1. \quad (1.9)$$

They proved that $\{x_n\}$ converges to the same point of F in a reflexive strictly Banach space with a uniformly Gâteaux differentiable norm.

In the similar way, Dhompongsa *et al.* [14] extended Browder's iteration to a strongly continuous semigroup of nonexpansive mappings $\{T(t) : t \geq 0\}$ in a complete CAT(0) space X as follows:

$$x_n = \alpha_n x_0 \oplus T(t_n)x_n, \quad \forall n \geq 1,$$

where C is a nonempty closed convex subset of a complete CAT(0) space X , $x_0 \in C$, $\{\alpha_n\}$ and $\{t_n\}$ are sequences of real numbers satisfying $0 < \alpha_n < 1$, $t_n > 0$, and $\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} \alpha_n/t_n = 0$. They proved that $\mathcal{F} \neq \emptyset$ and $\{x_n\}$ converges to the element of $\mathcal{F}$ nearest to u . For other related results, see [15, 16].

In 2012, Shi and Chen [17], studied the convergence theorems of the following Moudafi's viscosity iterations for a nonexpansive mapping T : for a contraction f on C and $t \in (0, 1)$, let $x_t \in C$ be a unique fixed point of the contraction $x \mapsto tf(x) \oplus (1-t)Tx$; i.e.,

$$x_t = tf(x_t) \oplus (1-t)Tx_t, \quad (1.10)$$

and $x_0 \in C$ is arbitrarily chosen and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n, \quad \forall n \geq 0, \quad (1.11)$$

where $\{\alpha_n\} \subset (0, 1)$. They proved $\{x_t\}$ defined by (1.10) converges strongly as $t \rightarrow 0$ to $\tilde{x} \in F(T)$ such that $\tilde{x} = P_{F(T)}f(\tilde{x})$ in the framework of CAT(0) space satisfying property $\mathcal{P}$, i.e., if for $x, u, y_1, y_2 \in X$,

$$d(x, P_{[x, y_1]}u)d(x, y_1) \leq d(x, P_{[x, y_2]}u)d(x, y_2) + d(x, u)d(y_1, y_2).$$

Furthermore, they also obtained that $\{x_n\}$ defined by (1.11) converges strongly as $n \rightarrow \infty$ to $\tilde{x} \in F(T)$ under certain appropriate conditions imposed on $\{\alpha_n\}$.

By using the concept of quasilinearization, Wangkeeree and Preechasilp [18] improved Shi and Chen's results. In fact, they proved the strong convergence theorems for two given iterative schemes (1.10) and (1.11) in a complete CAT(0) space without the property $\mathcal{P}$.

Motivated and inspired by Song and Xu [13], Dhompongsa *et al.* [14], and Wangkeeree and Preechasilp [18], in this paper we aim to study the strong convergence theorems of Moudafi's viscosity approximation methods for a one-parameter continuous semigroup of nonexpansive mappings $\mathcal{S} := \{T(t) : t \in \mathbb{R}^+\}$ in CAT(0) spaces. Let C be a nonempty, closed and convex subset of a CAT(0) space X . For a given contraction f on C and $\alpha_n \in (0, 1)$, let $x_n \in C$ be a unique fixed point of the contraction $x \mapsto \alpha_n f(x) \oplus (1-\alpha_n)T(t_n)x$; i.e.,

$$x_n = \alpha_n f(x_n) \oplus (1-\alpha_n)T(t_n)x_n, \quad n \geq 0, \quad (1.12)$$

and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T(t_n)x_n, \quad n \geq 0. \quad (1.13)$$

We prove that the iterative schemes $\{x_n\}$ defined by (1.12) and $\{x_n\}$ defined by (1.13) converge strongly to the same point $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}}f(\tilde{x})$, which is the unique solution of the variational inequality

$$\langle \overrightarrow{\tilde{x}f\tilde{x}}, \overrightarrow{\tilde{x}\tilde{x}} \rangle \geq 0, \quad x \in \mathcal{F},$$

where $\mathcal{F}$ is the common fixed point set of $\mathcal{S}$, that is,

$$\mathcal{F} := F(\mathcal{S}) = \{x \in C : T(t)x = x, t \in \mathbb{R}^+\} = \bigcap_{t \in \mathbb{R}^+} F(T(t)).$$

2 Preliminaries

In this paper, we write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z, x) = td(x, y) \quad \text{and} \quad d(z, y) = (1-t)d(x, y).$$

We also denote by $[x, y]$ the geodesic segment joining from x to y , that is, $[x, y] = \{(1-t)x \oplus ty : t \in [0, 1]\}$. A subset C of a CAT(0) space is convex if $[x, y] \subseteq C$ for all $x, y \in C$.

The following lemmas play an important role in our paper.

Lemma 2.1 [2, Proposition 2.2] *Let X be a CAT(0) space, $p, q, r, s \in X$ and $\lambda \in [0, 1]$. Then*

$$d(\lambda p \oplus (1-\lambda)q, \lambda r \oplus (1-\lambda)s) \leq \lambda d(p, r) + (1-\lambda)d(q, s).$$

Lemma 2.2 [19, Lemma 2.4] *Let X be a CAT(0) space, $x, y, z \in X$ and $\lambda \in [0, 1]$. Then*

$$d(\lambda x \oplus (1-\lambda)y, z) \leq \lambda d(x, z) + (1-\lambda)d(y, z).$$

Lemma 2.3 [19, Lemma 2.5] *Let X be a CAT(0) space, $x, y, z \in X$ and $\lambda \in [0, 1]$. Then*

$$d^2(\lambda x \oplus (1-\lambda)y, z) \leq \lambda d^2(x, z) + (1-\lambda)d^2(y, z) - \lambda(1-\lambda)d^2(x, y).$$

The concept of Δ -convergence introduced by Lim [20] in 1976 was shown by Kirk and Panyanak [21] in CAT(0) spaces to be very similar to the weak convergence in Banach space setting. Next, we give the concept of Δ -convergence and collect some basic properties.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X . For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from Proposition 7 of [22] that in a complete CAT(0) space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\} \subset X$ is said to Δ -converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. The uniqueness of an asymptotic center implies that a CAT(0) space X satisfies Opial's property, i.e., for given $\{x_n\} \subset X$ such that $\{x_n\}$ Δ -converges to x and given $y \in X$ with $y \neq x$,

$$\limsup_{n \rightarrow \infty} d(x_n, x) < \limsup_{n \rightarrow \infty} d(x_n, y).$$

Since it is not possible to formulate the concept of demiclosedness in a CAT(0) setting, as stated in linear spaces, let us formally say that ' $T - T$ is demiclosed at zero' if the conditions $\{x_n\} \subseteq C$ Δ -converges to x and $d(x_n, Tx_n) \rightarrow 0$ imply $x \in F(T)$.

Lemma 2.4 [21] *Every bounded sequence in a complete CAT(0) space always has a Δ -convergent subsequence.*

Lemma 2.5 [23] *If C is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C , then the asymptotic center of $\{x_n\}$ is in C .*

Lemma 2.6 [23] *If C is a closed convex subset of X and $T : C \rightarrow X$ is a nonexpansive mapping, then the conditions $\{x_n\}$ Δ -converges to x and $d(x_n, Tx_n) \rightarrow 0$ imply $x \in C$ and $Tx = x$.*

Having the notion of quasilinearization, Kakavandi and Amini [6] introduced the following notion of convergence.

A sequence $\{x_n\}$ in the complete CAT(0) space (X, d) w -converges to $x \in X$ if

$$\lim_{n \rightarrow \infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle = 0,$$

i.e., $\lim_{n \rightarrow \infty} (d^2(x_n, x) - d^2(x_n, y) + d^2(x, y)) = 0$ for all $y \in X$.

It is obvious that convergence in the metric implies w -convergence, and it is easy to check that w -convergence implies Δ -convergence [6, Proposition 2.5], but it is showed in [24, Example 4.7] that the converse is not valid. However, the following lemma shows another characterization of Δ -convergence as well as, more explicitly, a relation between w -convergence and Δ -convergence.

Lemma 2.7 [24, Theorem 2.6] *Let X be a complete CAT(0) space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ Δ -converges to x if and only if $\limsup_{n \rightarrow \infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle \leq 0$ for all $y \in X$.*

Lemma 2.8 [25, Lemma 2.1] *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the property*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\beta_n, \quad n \geq 0,$$

where $\{\alpha_n\} \subseteq (0, 1)$ and $\{\beta_n\} \subseteq \mathbb{R}$ such that

- (i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (ii) $\limsup_{n \rightarrow \infty} \beta_n \leq 0$ or $\sum_{n=0}^{\infty} |\alpha_n\beta_n| < \infty$.

Then $\{a_n\}$ converges to zero as $n \rightarrow \infty$.

3 Viscosity approximation methods

In this section, we present the strong convergence theorems of Moudafi's viscosity approximation methods for a one-parameter continuous semigroup of nonexpansive mappings $S := \{T(t) : t \in \mathbb{R}^+\}$ in CAT(0) spaces. Before proving main results, we need the following two vital lemmas.

Lemma 3.1 *Let X be a complete CAT(0) space. Then, for all $u, x, y \in X$, the following inequality holds:*

$$d^2(x, u) \leq d^2(y, u) + 2\langle \overrightarrow{xy}, \overrightarrow{xu} \rangle.$$

Proof Using (1.2), we have that

$$\begin{aligned} d^2(y, u) - d^2(x, u) - 2\langle \vec{y}\vec{x}, \vec{x}\vec{u} \rangle &= d^2(y, u) - d^2(x, u) - 2\langle \vec{y}\vec{u}, \vec{x}\vec{u} \rangle - 2\langle \vec{u}\vec{x}, \vec{x}\vec{u} \rangle \\ &= d^2(y, u) - d^2(x, u) - 2\langle \vec{y}\vec{u}, \vec{x}\vec{u} \rangle + 2d^2(x, u) \\ &= d^2(y, u) + d^2(x, u) - 2\langle \vec{y}\vec{u}, \vec{x}\vec{u} \rangle \\ &\geq d^2(y, u) + d^2(x, u) - 2d(y, u)d(x, u) \\ &= (d^2(y, u) - d^2(x, u))^2 \geq 0. \end{aligned}$$

Therefore we obtain that

$$d^2(x, u) \leq d^2(y, u) + 2\langle \vec{x}\vec{y}, \vec{x}\vec{u} \rangle,$$

which is the desired result. $\square$

Lemma 3.2 Let X be a CAT(0) space. For any $t \in [0, 1]$ and $u, v \in X$, let $u_t = tu \oplus (1-t)v$. Then, for all $x, y \in X$,

- (i) $\langle \vec{u}_t\vec{x}, \vec{u}_t\vec{y} \rangle \leq t\langle \vec{u}\vec{x}, \vec{u}\vec{y} \rangle + (1-t)\langle \vec{v}\vec{x}, \vec{u}\vec{y} \rangle$;
(ii) $\langle \vec{u}_t\vec{x}, \vec{u}\vec{y} \rangle \leq t\langle \vec{u}\vec{x}, \vec{u}\vec{y} \rangle + (1-t)\langle \vec{v}\vec{x}, \vec{u}\vec{y} \rangle$ and $\langle \vec{u}_t\vec{x}, \vec{v}\vec{y} \rangle \leq t\langle \vec{u}\vec{x}, \vec{v}\vec{y} \rangle + (1-t)\langle \vec{v}\vec{x}, \vec{v}\vec{y} \rangle$.

Proof (i) It follows from (CN)-inequality (1.1) that

$$\begin{aligned} 2\langle \vec{u}_t\vec{x}, \vec{u}_t\vec{y} \rangle &= d^2(u_t, y) + d^2(x, u_t) - d^2(x, y) \\ &\leq td^2(u, y) + (1-t)d^2(v, y) - t(1-t)d^2(u, v) + d^2(x, u_t) - d^2(x, y) \\ &= td^2(u, y) + td^2(x, u_t) - td^2(u, u_t) - td^2(x, y) \\ &\quad + (1-t)d^2(v, y) + (1-t)d^2(x, u_t) - (1-t)d^2(v, u_t) - (1-t)d^2(x, y) \\ &\quad + td^2(u, u_t) + (1-t)d^2(v, u_t) - t(1-t)d^2(u, v) \\ &= t[d^2(u, y) + d^2(x, u_t) - d^2(u, u_t) - d^2(x, y)] \\ &\quad + (1-t)[d^2(v, y) + d^2(x, u_t) - d^2(v, u_t) - d^2(x, y)] \\ &\quad + t(1-t)d^2(u, v) + (1-t)t^2d^2(u, v) - t(1-t)d^2(u, v) \\ &= t\langle \vec{u}\vec{x}, \vec{u}\vec{y} \rangle + (1-t)\langle \vec{v}\vec{x}, \vec{u}\vec{y} \rangle. \end{aligned}$$

(ii) The proof is similar to (i). $\square$

For any $\alpha_n \in (0, 1)$, $t_n \in [0, \infty)$ and a contraction f with coefficient $\alpha \in (0, 1)$, define the mapping $G_n : C \rightarrow C$ by

$$G_n(x) = \alpha_n f(x) \oplus (1 - \alpha_n)T(t_n)x, \quad \forall x \in C. \quad (3.1)$$

It is not hard to see that G_n is a contraction on C . Indeed, for $x, y \in C$, we have

$$\begin{aligned} d(G_n(x), G_n(y)) &= d(\alpha_n f(x) \oplus (1 - \alpha_n)T(t_n)x, \alpha_n f(y) \oplus (1 - \alpha_n)T(t_n)y) \\ &\leq d(\alpha_n f(x) \oplus (1 - \alpha_n)T(t_n)x, \alpha_n f(y) \oplus (1 - \alpha_n)T(t_n)x) \end{aligned}$$

$$\begin{aligned}
 &+ d(\alpha_n f(y) \oplus (1 - \alpha_n)T(t_n)x, \alpha_n f(y) \oplus (1 - \alpha_n)T(t_n)y) \\
 &\leq \alpha_n d(f(x), f(y)) + (1 - \alpha_n)d(T(t_n)x, T(t_n)y) \\
 &\leq \alpha_n \alpha d(x, y) + (1 - \alpha_n)d(x, y) \\
 &= (1 - \alpha_n(1 - \alpha))d(x, y).
 \end{aligned}$$

Therefore we have that G_n is a contraction mapping. Let $x_n \in C$ be the unique fixed point of G_n ; that is,

$$x_n = \alpha_n f(x_n) \oplus (1 - \alpha_n)T(t_n)x_n \quad \text{for all } n \geq 0. \quad (3.2)$$

Now we are in a position to state and prove our main results.

Theorem 3.3 *Let C be a closed convex subset of a complete CAT(0) space X , and let $\{T(t)\}$ be a one-parameter continuous semigroup of nonexpansive mappings on C satisfying $\mathcal{F} \neq \emptyset$ and uniformly asymptotically regular (in short, u.a.r.) on C , that is, for all $h \geq 0$ and any bounded subset B of C ,*

$$\lim_{t \rightarrow \infty} \sup_{x \in B} d(T(h)(T(t)x), T(t)x) = 0.$$

Let f be a contraction on C with coefficient $0 < \alpha < 1$. Suppose that $t_n \in [0, \infty)$, $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} t_n = \infty$, $\lim_{n \rightarrow \infty} \alpha_n = 0$ and let $\{x_n\}$ be given by (3.2). Then $\{x_n\}$ converges strongly as $n \rightarrow \infty$ to $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}}f(\tilde{x})$, which is equivalent to the following variational inequality:

$$\langle \overrightarrow{\tilde{x}f\tilde{x}}, \overrightarrow{\tilde{x}\tilde{x}} \rangle \geq 0, \quad \forall x \in \mathcal{F}. \quad (3.3)$$

Proof We first show that $\{x_n\}$ is bounded. For any $p \in \mathcal{F}$, we have that

$$\begin{aligned}
 d(x_n, p) &= d(\alpha_n f(x_n) \oplus (1 - \alpha_n)T(t_n)x_n, p) \leq \alpha_n d(f(x_n), p) + (1 - \alpha_n)d(T(t_n)x_n, p) \\
 &\leq \alpha_n d(f(x_n), p) + (1 - \alpha_n)d(x_n, p).
 \end{aligned}$$

Then

$$d(x_n, p) \leq d(f(x_n), p) \leq d(f(x_n), f(p)) + d(f(p), p) \leq \alpha d(x_n, p) + d(f(p), p).$$

This implies that

$$d(x_n, p) \leq \frac{1}{1 - \alpha} d(f(p), p).$$

Hence $\{x_n\}$ is bounded, so are $\{T(t_n)x_n\}$ and $\{f(x_n)\}$. We get that

$$\begin{aligned}
 d(x_n, T(t_n)x_n) &= d(\alpha_n f(x_n) \oplus (1 - \alpha_n)T(t_n)x_n, T(t_n)x_n) \\
 &\leq \alpha_n d(f(x_n), T(t_n)x_n) + (1 - \alpha_n)d(T(t_n)x_n, T(t_n)x_n) \\
 &\leq \alpha_n d(f(x_n), T(t_n)x_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned}$$

Since $\{T(t)\}$ is u.a.r. and $\lim_{n \rightarrow \infty} t_n = \infty$, then for all $h > 0$,

$$\lim_{n \rightarrow \infty} d(T(h)(T(t_n)x_n), T(t_n)x_n) \leq \lim_{n \rightarrow \infty} \sup_{x \in B} d(T(h)(T(t_n)x), T(t_n)x) = 0,$$

where B is any bounded subset of C containing $\{x_n\}$. Hence

$$\begin{aligned} d(x_n, T(h)x_n) &\leq d(x_n, T(t_n)x_n) + d(T(t_n)x_n, T(h)(T(t_n)x_n)) \\ &\quad + d(T(h)(T(t_n)x_n), T(h)x_n) \\ &\leq 2d(x_n, T(t_n)x_n) + d(T(t_n)x_n, T(h)(T(t_n)x_n)) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (3.4)$$

We will show that $\{x_n\}$ contains a subsequence converging strongly to $\tilde{x}$ such that $\tilde{x} = P_{F(T)}f(\tilde{x})$, which is equivalent to the following variational inequality:

$$\langle \overrightarrow{\tilde{x}f\tilde{x}}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in \mathcal{F}. \quad (3.5)$$

Since $\{x_n\}$ is bounded, by Lemma 2.4, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which Δ -converges to a point $\tilde{x}$, denoted by $\{x_j\}$. We claim that $\tilde{x} \in \mathcal{F}$. Since every $\text{CAT}(0)$ space has Opial's property, for any $h \geq 0$, if $T(h)\tilde{x} \neq \tilde{x}$, we have

$$\begin{aligned} \limsup_{j \rightarrow \infty} d(x_j, T(h)\tilde{x}) &\leq \limsup_{j \rightarrow \infty} \{d(x_j, T(h)x_j) + d(T(h)x_j, T(h)\tilde{x})\} \\ &\leq \limsup_{j \rightarrow \infty} \{d(x_j, T(h)x_j) + d(x_j, \tilde{x})\} \\ &= \limsup_{j \rightarrow \infty} d(x_j, \tilde{x}) \\ &< \limsup_{j \rightarrow \infty} d(x_j, T(h)\tilde{x}). \end{aligned}$$

This is a contradiction, and hence $\tilde{x} \in \mathcal{F}$. So we have the claim. It follows from Lemma 3.2(i) that

$$\begin{aligned} d^2(x_j, \tilde{x}) &= \langle \overrightarrow{x_j\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle \\ &\leq \alpha_j \langle \overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle + (1 - \alpha_j) \langle \overrightarrow{T(t_j)x_j\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle \\ &\leq \alpha_j \langle \overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle + (1 - \alpha_j) d(T(t_j)x_j, \tilde{x}) d(x_j, \tilde{x}) \\ &\leq \alpha_j \langle \overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle + (1 - \alpha_j) d^2(x_j, \tilde{x}). \end{aligned}$$

It follows that

$$\begin{aligned} d^2(x_j, \tilde{x}) &\leq \langle \overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle \\ &= \langle \overrightarrow{f(x_j)f(\tilde{x})}, \overrightarrow{x_j\tilde{x}} \rangle + \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle \\ &\leq d(f(x_j), f(\tilde{x})) d(x_j, \tilde{x}) + \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle \\ &\leq \alpha d^2(x_j, \tilde{x}) + \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle, \end{aligned}$$

and thus

$$d^2(x_j, \tilde{x}) \leq \frac{1}{1-\alpha} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle. \quad (3.6)$$

Since $\{x_j\}$ Δ -converges to $\tilde{x}$, by Lemma 2.7, we have

$$\limsup_{n \rightarrow \infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle \leq 0.$$

It follows from (3.6) that $\{x_j\}$ converges strongly to $\tilde{x}$. Next, we show that $\tilde{x}$ solves the variational inequality (3.3). Applying Lemma 2.3, for any $q \in \mathcal{F}$,

$$\begin{aligned} d^2(x_j, q) &= d^2(\alpha_j f(x_j) \oplus (1-\alpha_j)T(t_j)x_j, q) \\ &\leq \alpha_j d^2(f(x_j), q) + (1-\alpha_j)d^2(T(t_j)x_j, q) - \alpha_j(1-\alpha_j)d^2(f(x_j), T(t_j)x_j) \\ &\leq \alpha_j d^2(f(x_j), q) + (1-\alpha_j)d^2(x_j, q) - \alpha_j(1-\alpha_j)d^2(f(x_j), T(t_j)x_j). \end{aligned}$$

It implies that

$$d^2(x_j, q) \leq d^2(f(x_j), q) - (1-\alpha_j)d^2(f(x_j), T(t_j)x_j).$$

Taking the limit through $j \rightarrow \infty$, we can get that

$$d^2(\tilde{x}, q) \leq d^2(f(\tilde{x}), q) - d^2(f(\tilde{x}), \tilde{x}).$$

Hence

$$0 \leq \frac{1}{2} [d^2(\tilde{x}, \tilde{x}) + d^2(f(\tilde{x}), q) - d^2(\tilde{x}, q) - d^2(f(\tilde{x}), \tilde{x})] = \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{q\tilde{x}} \rangle, \quad \forall q \in \mathcal{F}.$$

That is, $\tilde{x}$ solves the inequality (3.3). Finally, we show that the sequence $\{x_n\}$ converges to $\tilde{x}$. Assume that $x_{n_i} \rightarrow \hat{x}$, where $i \rightarrow \infty$. By the same argument, we get that $\hat{x} \in \mathcal{F}$ and solves the variational inequality (3.3), i.e.,

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle \leq 0, \quad (3.7)$$

and

$$\langle \overrightarrow{\hat{x}f(\hat{x})}, \overrightarrow{\hat{x}\tilde{x}} \rangle \leq 0. \quad (3.8)$$

Adding up (3.7) and (3.8), we get that

$$\begin{aligned} 0 &\geq \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle - \langle \overrightarrow{\hat{x}f(\hat{x})}, \overrightarrow{\hat{x}\tilde{x}} \rangle \\ &= \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle + \langle \overrightarrow{f(\hat{x})f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle - \langle \overrightarrow{\tilde{x}\hat{x}}, \overrightarrow{\tilde{x}\hat{x}} \rangle - \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle \\ &= \langle \overrightarrow{\tilde{x}\hat{x}}, \overrightarrow{\tilde{x}\hat{x}} \rangle - \langle \overrightarrow{f(\hat{x})f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle \\ &\geq \langle \overrightarrow{\tilde{x}\hat{x}}, \overrightarrow{\tilde{x}\hat{x}} \rangle - d(f(\hat{x}), f(\tilde{x}))d(\hat{x}, \tilde{x}) \end{aligned}$$

$$\begin{aligned} &\geq d^2(\tilde{x}, \hat{x}) - \alpha d(\hat{x}, \tilde{x})d(\hat{x}, \tilde{x}) \\ &\geq d^2(\tilde{x}, \hat{x}) - \alpha d^2(\hat{x}, \tilde{x}) \\ &\geq (1 - \alpha)d^2(\tilde{x}, \hat{x}). \end{aligned}$$

Since $0 < \alpha < 1$, we have that $d(\tilde{x}, \hat{x}) = 0$, and so $\tilde{x} = \hat{x}$. Hence the sequence x_n converges strongly to $\tilde{x}$, which is the unique solution to the variational inequality (3.3). This completes the proof. $\square$

If $f \equiv u$, then the following result can be obtained directly from Theorem 3.3.

Corollary 3.4 *Let C be a closed convex subset of a complete CAT(0) space X , and let $\{T(t)\}$ be a one-parameter continuous semigroup of nonexpansive mappings on C satisfying $\mathcal{F} \neq \emptyset$ and uniformly asymptotically regular (in short, u.a.r.) on C , that is, for all $h \geq 0$ and any bounded subset B of C ,*

$$\lim_{t \rightarrow \infty} \sup_{x \in B} d(T(h)(T(t)x), T(t)x) = 0.$$

Let u be any element in C . Suppose $t_n \in [0, \infty)$, $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} t_n = \infty$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$ and let $\{x_n\}$ be given by

$$x_n = \alpha_n u \oplus (1 - \alpha_n)T(t_n)x_n.$$

Then $\{x_n\}$ converges strongly as $n \rightarrow \infty$ to $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}}\tilde{x}$, which is equivalent to the following variational inequality:

$$\langle \vec{xu}, \vec{x\tilde{x}} \rangle \geq 0, \quad x \in \mathcal{F}. \quad (3.9)$$

Theorem 3.5 *Let C be a closed convex subset of a complete CAT(0) space X , and let $\{T(t)\}$ be a one-parameter continuous semigroup of nonexpansive mappings on C satisfying $\mathcal{F} \neq \emptyset$ and uniformly asymptotically regular (in short, u.a.r.) on C , that is, for all $h \geq 0$ and any bounded subset B of C ,*

$$\lim_{t \rightarrow \infty} \sup_{x \in B} d(T(h)(T(t)x), T(t)x) = 0.$$

Let f be a contraction on C with coefficient $0 < \alpha < 1$. Suppose that $t_n \in [0, \infty)$, $\alpha_n \in (0, 1)$, $x_0 \in C$, and $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n)T(t_n)x_n, \quad \forall n \geq 0, \quad (3.10)$$

where $\{\alpha_n\} \subset (0, 1)$ satisfies the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and
- (iii) $\lim_{n \rightarrow \infty} t_n = \infty$.

Then $\{x_n\}$ converges strongly as $n \rightarrow \infty$ to $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}}f(\tilde{x})$, which is equivalent to the variational inequality (3.3).

Proof We first show that the sequence $\{x_n\}$ is bounded. For any $p \in \mathcal{F}$, we have that

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n f(x_n) \oplus (1 - \alpha_n)T(t_n)x_n, p) \\ &\leq \alpha_n d(f(x_n), p) + (1 - \alpha_n)d(T(t_n)x_n, p) \\ &\leq \alpha_n (d(f(x_n), f(p)) + d(f(p), p)) + (1 - \alpha_n)d(T(t_n)x_n, p) \\ &\leq \max \left\{ d(x_n, p), \frac{1}{1 - \alpha} d(f(p), p) \right\}. \end{aligned}$$

By induction, we have

$$d(x_n, p) \leq \max \left\{ d(x_0, p), \frac{1}{1 - \alpha} d(f(p), p) \right\}$$

for all $n \in \mathbb{N}$. Hence $\{x_n\}$ is bounded, so are $\{T(t_n)x_n\}$ and $\{f(x_n)\}$. Using the assumption that $\lim_{n \rightarrow \infty} \alpha_n = 0$, we get that

$$d(x_{n+1}, T(t_n)x_n) \leq \alpha_n d(f(x_n), T(t_n)x_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since $\{T(t)\}$ is u.a.r. and $\lim_{n \rightarrow \infty} t_n = \infty$, then for all $h \geq 0$,

$$\lim_{n \rightarrow \infty} d(T(h)(T(t_n)x_n), T(t_n)x_n) \leq \lim_{n \rightarrow \infty} \sup_{x \in B} d(T(h)(T(t_n)x), T(t_n)x) = 0,$$

where B is any bounded subset of C containing $\{x_n\}$. Hence

$$\begin{aligned} d(x_{n+1}, T(h)x_{n+1}) &\leq d(x_{n+1}, T(t_n)x_n) + d(T(t_n)x_n, T(h)(T(t_n)x_n)) \\ &\quad + d(T(h)(T(t_n)x_n), T(h)x_{n+1}) \\ &\leq 2d(x_{n+1}, T(t_n)x_n) + d(T(t_n)x_n, T(h)(T(t_n)x_n)) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (3.11)$$

Let $\{z_m\}$ be a sequence in C such that

$$z_m = \alpha_m f(z_m) \oplus (1 - \alpha_m)T(t_m)z_m.$$

It follows from Theorem 3.3 that $\{z_m\}$ converges strongly as $m \rightarrow \infty$ to a fixed point $\tilde{x} \in \mathcal{F}$, which solves the variational inequality (3.3). Now, we claim that

$$\limsup_{n \rightarrow \infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \leq 0.$$

It follows from Lemma 3.2(i) that

$$\begin{aligned} d^2(z_m, x_{n+1}) &= \langle \overrightarrow{z_m x_{n+1}}, \overrightarrow{z_m x_{n+1}} \rangle \\ &\leq \alpha_m \langle \overrightarrow{f(z_m)x_{n+1}}, \overrightarrow{z_m x_{n+1}} \rangle + (1 - \alpha_m) \langle \overrightarrow{T(t_m)z_m x_{n+1}}, \overrightarrow{z_m x_{n+1}} \rangle \\ &= \alpha_m \langle \overrightarrow{f(z_m)f(\tilde{x})}, \overrightarrow{z_m x_{n+1}} \rangle + \alpha_m \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_{n+1}} \rangle + \alpha_m \langle \overrightarrow{\tilde{x}z_m}, \overrightarrow{z_m x_{n+1}} \rangle \end{aligned}$$

$$\begin{aligned}
 & + \alpha_m \langle \overrightarrow{z_m x_{n+1}}, \overrightarrow{z_m x_{n+1}} \rangle + (1 - \alpha_m) \langle \overrightarrow{T(t_m) z_m T(t_m) x_{n+1}}, \overrightarrow{z_m x_{n+1}} \rangle \\
 & + (1 - \alpha_m) \langle \overrightarrow{T(t_m) x_{n+1} x_{n+1}}, \overrightarrow{z_m x_{n+1}} \rangle \\
 & \leq \alpha_m \alpha d(z_m, \tilde{x}) d(z_m, x_{n+1}) + \alpha_m \langle \overrightarrow{f(\tilde{x}) \tilde{x}}, \overrightarrow{z_m x_{n+1}} \rangle + \alpha_m d(\tilde{x}, z_m) d(z_m, x_{n+1}) \\
 & + \alpha_m d^2(z_m, x_{n+1}) + (1 - \alpha_m) d^2(z_m, x_{n+1}) \\
 & + (1 - \alpha_m) d(T(t_m) x_{n+1}, x_{n+1}) d(z_m, x_{n+1}) \\
 & \leq \alpha_m \alpha d(z_m, \tilde{x}) M + \alpha_m \langle \overrightarrow{f(\tilde{x}) \tilde{x}}, \overrightarrow{z_m x_{n+1}} \rangle + \alpha_m d(\tilde{x}, z_m) M + \alpha_m d^2(z_m, x_{n+1}) \\
 & + (1 - \alpha_m) d^2(z_m, x_{n+1}) + (1 - \alpha_m) d(T(t_m) x_{n+1}, x_{n+1}) M \\
 & \leq d^2(z_m, x_{n+1}) + \alpha_m \alpha d(z_m, \tilde{x}) M + \alpha_m d(\tilde{x}, z_m) M + d(T(t_m) x_{n+1}, x_{n+1}) M \\
 & + \alpha_m \langle \overrightarrow{f(\tilde{x}) \tilde{x}}, \overrightarrow{z_m x_{n+1}} \rangle,
 \end{aligned}$$

where $M \geq \sup_{m,n \geq 1} \{d(z_m, x_n)\}$. This implies that

$$\langle \overrightarrow{f(\tilde{x}) \tilde{x}}, \overrightarrow{x_{n+1} z_m} \rangle \leq (1 + \alpha) d(z_m, \tilde{x}) M + \frac{d(T(t_m) x_{n+1}, x_{n+1})}{\alpha_m} M. \quad (3.12)$$

Taking the upper limit as $n \rightarrow \infty$ first, and then $m \rightarrow \infty$, inequality (3.12) yields that

$$\limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \langle \overrightarrow{f(\tilde{x}) \tilde{x}}, \overrightarrow{x_{n+1} z_m} \rangle \leq 0. \quad (3.13)$$

Since

$$\begin{aligned}
 \langle \overrightarrow{f(\tilde{x}) \tilde{x}}, \overrightarrow{x_{n+1} \tilde{x}} \rangle & = \langle \overrightarrow{f(\tilde{x}) \tilde{x}}, \overrightarrow{x_{n+1} z_m} \rangle + \langle \overrightarrow{f(\tilde{x}) \tilde{x}}, \overrightarrow{z_m \tilde{x}} \rangle \\
 & \leq \langle \overrightarrow{f(\tilde{x}) \tilde{x}}, \overrightarrow{x_{n+1} z_m} \rangle + d(f(\tilde{x}), \tilde{x}) d(z_m, \tilde{x}).
 \end{aligned}$$

Thus, by taking the upper limit as $n \rightarrow \infty$ first, and then $m \rightarrow \infty$ the last inequality, it follows from $z_m \rightarrow \tilde{x}$ and (3.13) that

$$\limsup_{n \rightarrow \infty} \langle \overrightarrow{f(\tilde{x}) \tilde{x}}, \overrightarrow{x_{n+1} \tilde{x}} \rangle \leq 0.$$

Finally, we prove that $x_n \rightarrow \tilde{x}$ as $n \rightarrow \infty$. For any $n \in \mathbb{N}$, we set $y_n = \alpha_n \tilde{x} \oplus (1 - \alpha_n) T(t_n) x_n$. It follows from Lemma 3.1 and Lemma 3.2(i), (ii) that

$$\begin{aligned}
 d^2(x_{n+1}, \tilde{x}) & \leq d^2(y_n, \tilde{x}) + 2 \langle \overrightarrow{x_{n+1} y_n}, \overrightarrow{x_{n+1} \tilde{x}} \rangle \\
 & \leq (\alpha_n d(\tilde{x}, \tilde{x}) + (1 - \alpha_n) d(T(t_n) x_n, \tilde{x}))^2 \\
 & + 2 [\alpha_n \langle \overrightarrow{f(x_n) y_n}, \overrightarrow{x_{n+1} \tilde{x}} \rangle + (1 - \alpha_n) \langle \overrightarrow{T(t_n) x_n y_n}, \overrightarrow{x_{n+1} \tilde{x}} \rangle] \\
 & \leq (1 - \alpha_n)^2 d^2(x_n, \tilde{x}) + 2 [\alpha_n \alpha_n \langle \overrightarrow{f(x_n) \tilde{x}}, \overrightarrow{x_{n+1} \tilde{x}} \rangle + \alpha_n (1 - \alpha_n) \langle \overrightarrow{f(x_n) T(t_n) x_n}, \overrightarrow{x_{n+1} \tilde{x}} \rangle] \\
 & + (1 - \alpha_n) \alpha_n \langle \overrightarrow{T(t_n) x_n \tilde{x}}, \overrightarrow{x_{n+1} \tilde{x}} \rangle + (1 - \alpha_n) (1 - \alpha_n) \langle \overrightarrow{T(t_n) x_n T(t_n) x_n}, \overrightarrow{x_{n+1} \tilde{x}} \rangle] \\
 & \leq (1 - \alpha_n)^2 d^2(x_n, \tilde{x}) + 2 [\alpha_n \alpha_n \langle \overrightarrow{f(x_n) \tilde{x}}, \overrightarrow{x_{n+1} \tilde{x}} \rangle + \alpha_n (1 - \alpha_n) \langle \overrightarrow{f(x_n) T(t_n) x_n}, \overrightarrow{x_{n+1} \tilde{x}} \rangle] \\
 & + (1 - \alpha_n) \alpha_n \langle \overrightarrow{T(t_n) x_n \tilde{x}}, \overrightarrow{x_{n+1} \tilde{x}} \rangle + (1 - \alpha_n)^2 d(T(t_n) x_n, T(t_n) x_n) d(x_{n+1}, \tilde{x})
 \end{aligned}$$

$$\begin{aligned}
 &= (1 - \alpha_n)^2 d^2(x_n, \tilde{x}) + 2[\alpha_n^2 \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + \alpha_n(1 - \alpha_n) \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle] \\
 &= (1 - \alpha_n)^2 d^2(x_n, \tilde{x}) + 2\alpha_n \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \\
 &= (1 - \alpha_n)^2 d^2(x_n, \tilde{x}) + 2\alpha_n \langle \overrightarrow{f(x_n)f(\tilde{x})}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + 2\alpha_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \\
 &\leq (1 - \alpha_n)^2 d^2(x_n, \tilde{x}) + 2\alpha_n \alpha d(x_n, \tilde{x}) d(x_{n+1}, \tilde{x}) + 2\alpha_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \\
 &\leq (1 - \alpha_n)^2 d^2(x_n, \tilde{x}) + \alpha_n \alpha (d^2(x_n, \tilde{x}) + d^2(x_{n+1}, \tilde{x})) + 2\alpha_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle,
 \end{aligned}$$

which implies that

$$\begin{aligned}
 d^2(x_{n+1}, \tilde{x}) &\leq \frac{1 - (2 - \alpha)\alpha_n + \alpha_n^2}{1 - \alpha\alpha_n} d^2(x_n, \tilde{x}) + \frac{2\alpha_n}{1 - \alpha\alpha_n} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \\
 &\leq \frac{1 - (2 - \alpha)\alpha_n}{1 - \alpha\alpha_n} d^2(x_n, \tilde{x}) + \frac{2\alpha_n}{1 - \alpha\alpha_n} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + \alpha_n^2 M,
 \end{aligned}$$

where $M \geq \sup_{n \geq 0} \{d^2(x_n, \tilde{x})\}$. It then follows that

$$d^2(x_{n+1}, \tilde{x}) \leq (1 - \alpha'_n) d^2(x_n, \tilde{x}) + \alpha'_n \beta'_n,$$

where

$$\alpha'_n = \frac{2(1 - \alpha)\alpha_n}{1 - \alpha\alpha_n} \quad \text{and} \quad \beta'_n = \frac{(1 - \alpha\alpha_n)\alpha_n}{2(1 - \alpha)} M + \frac{1}{(1 - \alpha)} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle.$$

Applying Lemma 2.8, we can conclude that $x_n \rightarrow \tilde{x}$. This completes the proof. $\square$

If $f \equiv u$, then the following corollary can be obtained directly from Theorem 3.5.

Corollary 3.6 *Let C be a closed convex subset of a complete CAT(0) space X , and let $\{T(t)\}$ be a one-parameter continuous semigroup of nonexpansive mappings on C satisfying $\mathcal{F} \neq \emptyset$ and uniformly asymptotically regular (in short, u.a.r.) on C , that is, for all $h \geq 0$ and any bounded subset B of C ,*

$$\lim_{t \rightarrow \infty} \sup_{x \in B} d(T(h)(T(t)x), T(t)x) = 0.$$

Suppose that $t_n \in [0, \infty)$, $\alpha_n \in (0, 1)$, $x_0 \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) T(t_n) x_n, \quad \forall n \geq 0, \quad (3.14)$$

where $\{\alpha_n\} \subset (0, 1)$ satisfies the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and
- (iii) $\lim_{n \rightarrow \infty} t_n = \infty$.

Then $\{x_n\}$ converges strongly as $n \rightarrow \infty$ to $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}} \tilde{x}$, which is equivalent to the variational inequality (3.9).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors read and approved the final manuscript.

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