

Research Article

Existence Theorems for Quasivariational Inequality Problem on Proximally Smooth Sets

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Received 1 November 2012; Accepted 23 December 2012

Academic Editor: Pavel Kurasov

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The concept of quasivariational inequality problem on proximally smooth sets is studied. Some sufficient conditions for solving the existence of solutions of such a problem are provided; also some interesting cases are discussed. Of course, due to the significance of proximally smooth sets, the results which are presented in this paper improve and extend many important results in the literature.

1. Introduction and Preliminaries

Variational inequality theory is a branch of the mathematics which is important, and it was also the inspiration for researchers to find new works, both in terms of mathematics and applications such as in economics, physical, biological, and engineering science, and other applied sciences. In 1973, Bensoussan et al. [1] introduced and studied the concept of the quasivariational inequality, which is a generalized form of the classical variational inequality that introduced by Stampacchia [2]. Later, many researchers proposed and analyzed the concept of the generalized quasivariational inequality; see, for example [3–18]. It is worth mentioning that the quasivariational inequality problem is of interest to study, since in many important problems the considered set also depends upon the solutions explicitly or implicitly. In fact, the concept of the quasi variational inequality has been applied in many fields such as in economics and transportation equilibrium, control and optimization theory, mathematical programming, and game theory.

In the early period of the research, it should be pointed out that almost all the results regarding the existence and iterative schemes for solving those variational inequalities problems are being considered in the convexity setting. This is because they need the convexity assumption for guaranteeing the well definedness of the proposed iterative algorithm which depends on the projection mapping. However, in fact, the convexity assumption may not be required because it

may be well defined even if the considered set is nonconvex (e.g., when the considered set is a closed subset of a finite dimensional space or a compact subset of a Hilbert space, etc.). However, it may be from the practical point of view one may see that the nonconvex problems are more useful than convex case. Consequently, now many researchers are paying attention to many nonconvex cases.

Let $T : H \rightarrow H$ be a mapping and let $C : H \rightarrow \mathcal{C}(H)$ be a set-valued mapping, where $\mathcal{C}(H)$ is a family of all nonempty closed subsets of H . In this paper, we are interested in the following problem: find $x^* \in C(x^*)$ such that

$$-Tx^* \in N_{C(x^*)}^P(x^*), \quad (1)$$

where $N_{C(x)}^P(x)$ is denoted for the proximal normal cone of $C(x)$ of x . The problem of type (1) was introduced by Bounkhel et al. [19]. In such a paper, they proposed some iterative algorithms for finding a solution of type (1), when the considered mapping is a set-valued mapping, while, in this paper, we will provide sufficient conditions for the existence of a solution of such a problem (1). To do this, let us start by recalling some basic concepts and useful results that will be needed in this work.

Let H be a real Hilbert space equipped with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. Let 2^H be denoted for the class of all nonempty subsets of H and $\mathcal{C}(H)$ denoted for the family of all nonempty closed subsets of H .

2. Main Results

In this paper, we are interested in the following classes of mappings.

Definition 8. Let $T : H \rightarrow H$ be a mapping. Then T is called

(a) β -strongly monotone if there exists $\beta > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \geq \beta \|x - y\|^2, \quad \forall x, y \in H, \quad (12)$$

(b) ξ -Lipschitz if there exists $\xi > 0$ such that

$$\|T(x) - T(y)\| \leq \xi \|x - y\|, \quad \forall x, y \in H. \quad (13)$$

That is, in other word, we will make the following assumption.

Assumption (A). Let $T : H \rightarrow H$ and $C : H \rightarrow [Cl(H)]_r$ be mappings.

(i) T is a β -strongly monotone and a ξ -Lipschitz single-valued mapping;

(ii) C is a κ -Lipschitz set-valued mapping;

(iii) there is $\omega \in [0, 1)$ such that

$$\|\text{Proj}_{C(x)}(z) - \text{Proj}_{C(y)}(z)\| \leq \omega \|x - y\|, \quad \forall x, y, z \in H. \quad (14)$$

Remark 9. Let $H = (-\infty, \infty)$ and $K = [a, b] \cup [c, d]$, for some positive real numbers a, b, c, d with $a < b < c < d$. If we define $C : H \rightarrow 2^H$ by $C(x) = m(x) + K$, where $m : H \rightarrow H$ is a mapping defined by $m(x) = kx$ for $k \in \mathbb{R}$, then we see that C is a $\max\{d - b, c - a\}$ -Lipschitz mapping, and Assumption (A) (iii) is satisfied with a constant $2(\max\{d - b, c - a\})$. This means that Assumptions (A) (ii) and (iii) are independent.

The following remark is very useful in order to prove our results. Before seeing that, for the sake of simplicity, let us make a notation: for each $r \in (0, \infty)$ and $s \in (0, r)$, we will write $t_s := r/(r - s)$.

Remark 10. Let r, β, ξ, ω , and δ be five positive real numbers such that $\beta \in (0, \xi)$, $\delta \in (0, r\sqrt{\beta^2 - \xi^2(1 - (1 - \omega)^2)/(1 - \omega)^2})$ and $\omega \in [0, 1 - \xi r\sqrt{\xi^2 - \beta^2}/(\xi^2 r - \beta\delta))$. If $h : [1, \diamond] \rightarrow [0, \infty)$ is a function defined by

$$h(t) = \frac{r(t - 1)}{t\delta} + \frac{\sqrt{\beta^2 t^2 - \xi^2 [t^2 - (1 - \omega)^2]}}{t\xi^2} \quad (15)$$

for all $t \in [1, \diamond]$, where $\diamond = \xi(1 - \omega)/\sqrt{\xi^2 - \beta^2}$, then h is an increasing continuous function on its domain. Moreover, we can check that β/ξ^2 is an element of the range of h .

Next, for a fixed positive real number $\kappa \in [0, 1/(2 \diamond - 1))$, we pick a real number $\zeta = (1 - \diamond\kappa)/\kappa(\diamond - 1)$. Here, we notice that $\zeta \in (1, \infty)$. Let us consider now a case when $h^{-1}(\beta/\xi^2) \in$

$[1, (1 + \zeta\kappa)/\kappa(1 + \zeta)]$. Then, by the definition of t_s and h is an increasing continuous function, for $s^* = r(1 - \kappa)/(1 + \zeta\kappa)$ we have

$$\frac{\beta}{\xi^2} < \frac{r(t_{s^*} - 1)}{t_{s^*}\delta} + \frac{\sqrt{\beta^2 t_{s^*}^2 - \xi^2 [t_{s^*}^2 - (1 - \omega)^2]}}{t_{s^*}\xi^2}, \quad (16)$$

that is,

$$\frac{\beta}{\xi^2} - \frac{\sqrt{\beta^2 t_{s^*}^2 - \xi^2 [t_{s^*}^2 - (1 - \omega)^2]}}{t_{s^*}\xi^2} < \frac{r(t_{s^*} - 1)}{t_{s^*}\delta}. \quad (17)$$

Since $r(t_{s^*} - 1)/t_{s^*} = s^*$, this gives

$$\frac{\beta}{\xi^2} - \frac{\sqrt{\beta^2 t_{s^*}^2 - \xi^2 [t_{s^*}^2 - (1 - \omega)^2]}}{t_{s^*}\xi^2} < \frac{s^*}{\delta}. \quad (18)$$

This allows us to take a real number ρ such that

$$\frac{\beta}{\xi^2} - f(t_{s^*}) < \rho < \min \left\{ \frac{\beta}{\xi^2} + f(t_{s^*}), \frac{s^*}{\delta} \right\}, \quad (19)$$

where $f(t) = \sqrt{\beta^2 t^2 - \xi^2 [t^2 - (1 - \omega)^2]}/t\xi^2$.

Now we are in a position to present our main results.

Theorem 11. Let $T : H \rightarrow H$ be a single-valued mapping and let $C : H \rightarrow [Cl(H)]_r$ be a set-valued mapping. Assume that Assumption (A) holds and the following control conditions are satisfied:

(i) $\delta \in (0, r\sqrt{\beta^2 - \xi^2(1 - (1 - \omega)^2)/(1 - \omega)^2})$ and $\omega \in [0, 1 - \xi r\sqrt{\xi^2 - \beta^2}/(\xi^2 r - \beta\delta))$, where $\delta = \sup\{\|T(x)\| : x \in H\}$;

(ii) $\kappa \in [0, 1/(2 \diamond - 1))$, where $\diamond = \xi(1 - \omega)/\sqrt{\xi^2 - \beta^2}$;

(iii) $h^{-1}(\beta/\xi^2) \in [1, (1 + \zeta\kappa)/\kappa(1 + \zeta)]$, where h and ζ are defined as Remark 10.

If there is $x_0 \in H$ such that $d(x_0, C(x_0)) < s^* - \rho\|Tx_0\|$, where $s^* = r(1 - \kappa)/(1 + \zeta\kappa)$ and ρ is a real number corresponding to s^* which is chosen as in (19), then the problem (1) has a solution.

Proof. Firstly, we will define a sequence $\{x_n\}$ in H as follows: consider an element x_0 in H such that $d(x_0, C(x_0)) < s^* - \rho\|Tx_0\|$; we see that

$$d(x_0 - \rho Tx_0, C(x_0)) \leq d(x_0, C(x_0)) + \rho\|Tx_0\| < s^*. \quad (20)$$

Write $\lambda := r((1 + \kappa)/(1 + \zeta\kappa))$. From the previous argument, we see that the sequence $\{x_n\}$ also has a property that

$$x_n - \rho T x_n \in [C(x_n)]_\lambda, \quad (30)$$

for all $n \in \mathbb{N}$.

Next, we will show that $\{x_n\}$ is a convergent sequence, and its limit is nothing but a solution of the problem (1).

Now, by using the Assumption (A) (iii) and Lemma 4 (ii), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\text{Proj}_{C(x_n)}(x_n - \rho T x_n) - \text{Proj}_{C(x_{n-1})}(x_{n-1} - \rho T x_{n-1})\| \\ &\leq \|\text{Proj}_{C(x_n)}(x_n - \rho T x_n) - \text{Proj}_{C(x_n)}(x_{n-1} - \rho T x_{n-1})\| \\ &\quad + \|\text{Proj}_{C(x_n)}(x_{n-1} - \rho T x_{n-1}) \\ &\quad - \text{Proj}_{C(x_{n-1})}(x_{n-1} - \rho T x_{n-1})\| \\ &\leq t_\lambda \|x_n - \rho T x_n - x_{n-1} + \rho T x_{n-1}\| + \omega \|x_n - x_{n-1}\|. \end{aligned} \quad (31)$$

Meanwhile, since T is a β -strongly monotone and ξ -Lipschitz mapping, we see that

$$\begin{aligned} \|x_n - \rho T x_n - x_{n-1} + \rho T x_{n-1}\|^2 &= \|x_n - x_{n-1} - \rho[T x_n - T x_{n-1}]\|^2 \\ &= \|x_n - x_{n-1}\|^2 - 2\rho \langle T x_n - T x_{n-1}, x_n - x_{n-1} \rangle \\ &\quad + \rho^2 \|T x_n - T x_{n-1}\|^2 \\ &\leq \|x_n - x_{n-1}\|^2 - 2\rho\beta \|x_n - x_{n-1}\|^2 \\ &\quad + \rho^2 \xi^2 \|x_n - x_{n-1}\|^2 \\ &= (1 - 2\rho\beta + \rho^2 \xi^2) \|x_n - x_{n-1}\|^2. \end{aligned} \quad (32)$$

Let $\theta_\rho = \max\{1 - 2\rho\beta + \rho^2 \xi^2, ((1 - \omega)(r - \lambda)/2r)^2\}$. Observe that $\theta_\rho \in [0, \infty)$. Moreover, it follows that

$$\|x_n - \rho T x_n - x_{n-1} + \rho T x_{n-1}\| \leq \sqrt{\theta_\rho} \|x_n - x_{n-1}\|. \quad (33)$$

By replacing (33) into (31), we get

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq t_\lambda \sqrt{\theta_\rho} \|x_n - x_{n-1}\| + \omega \|x_n - x_{n-1}\| \\ &= [t_\lambda \sqrt{\theta_\rho} + \omega] \|x_n - x_{n-1}\|. \end{aligned} \quad (34)$$

Let $\gamma := t_\lambda \sqrt{\theta_\rho} + \omega$. Then, by a choice of ρ , one can check that $\gamma < 1$. Subsequently, by (34), we have

$$\|x_{n+1} - x_n\| \leq \gamma^n \|x_1 - x_0\|, \quad \forall n \in \mathbb{N}. \quad (35)$$

Hence, for any $m \geq n$, we see that

$$\begin{aligned} \|x_m - x_n\| &\leq \sum_{i=n}^{m-1} \|x_{i+1} - x_i\| \\ &\leq \|x_1 - x_0\| \sum_{i=n}^{m-1} \gamma^i \\ &\leq \|x_1 - x_0\| \sum_{i=n}^{\infty} \gamma^i \\ &\leq \left(\frac{\gamma^n}{1 - \gamma} \right) \|x_1 - x_0\|. \end{aligned} \quad (36)$$

This implies that $\{x_n\}$ is a Cauchy sequence in H , since $\gamma < 1$. So, by the completeness of H , there exists $x^* \in H$ such that $x_n \rightarrow x^*$ as $n \rightarrow \infty$.

We now finish the proof by showing that x^* is a solution of problem (1). To do this, we will start by asserting that $x^* \in C(x^*)$. Indeed, since $x_{n+1} \in C(x_n)$ and C is a κ -Lipschitz mapping, we have

$$\begin{aligned} d(x^*, C(x^*)) &= |d(x_{n+1}, C(x_n)) - d(x^*, C(x^*))| \\ &\leq \|x_{n+1} - x^*\| + \kappa \|x_n - x^*\|. \end{aligned} \quad (37)$$

Thus, since $x_n \rightarrow x^*$, we see that $d(x^*, C(x^*)) = 0$. So, by the closedness of $C(x^*)$, it guarantees that $x^* \in C(x^*)$.

Next we show that $T x^* \in -N(C(x^*), x^*)$. Let us observe that

$$x_{n+1} \in \text{Proj}_{C(x_n)}(x_n - \rho T x_n), \quad \forall n \in \mathbb{N} \quad (38)$$

is equivalent to

$$x_{n+1} \in \text{Proj}_{C(x_n)}(x_{n+1} + (x_n - x_{n+1}) - \rho T x_n). \quad (39)$$

And this means

$$(x_n - x_{n+1}) - \rho T x_n \in N(C(x_n), x_{n+1}), \quad \forall n \in \mathbb{N}. \quad (40)$$

Thus, by using the continuity of mapping T , from Lemma 7 we see that $\rho T x^* \in -N(C(x^*), x^*)$. This implies $T x^* \in -N(C(x^*), x^*)$, as required. This means that x^* is a solution of the problem (1), and the proof is completed. $\square$

Remark 12. A condition which has been proposed in the assumptions of Theorem 11 is that “there is $x_0 \in H$ such that $d(x_0, C(x_0)) < s^* - \rho \|T x_0\|$, where $s^* = r(1 - \kappa)/(1 + \zeta\kappa)$.” Here, in view of Remark 10 together with the following facts, one may see that our choice $s^* = r(1 - \kappa)/(1 + \zeta\kappa)$ should be sharpest.

- (i) $s = r(1 - \kappa)/(1 + \zeta\kappa) \Leftrightarrow t_s = (1 + \zeta\kappa)/\kappa(1 + \zeta)$.
- (ii) The function $\zeta \mapsto (1 + \zeta\kappa)/\kappa(1 + \zeta)$ is an increasing function on its domain.
- (iii) $(1 + \zeta\kappa)/\kappa(1 + t) = \diamond \Leftrightarrow \zeta = (1 - \diamond\kappa)/\kappa(\diamond - 1)$, where $\diamond = \xi(1 - \omega)/\sqrt{\xi^2 - \beta^2}$.

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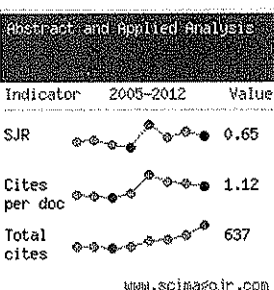
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