



On Gamma-Primary and Quasi Gamma-Primary Ideals in Gamma-AG-Groupoids

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Abstract

The purpose of the paper was to introduce the notion of a Γ -primary ideals in Γ -AG-groupoids, we studied Γ -ideals, Γ -primary and quasi Γ -primary ideals in Γ -AG-groupoids. Some characterizations of Γ -primary and quasi Γ -primary ideals were obtained. Moreover, we investigated relationships between Γ -primary and quasi Γ -primary ideals in Γ -AG-groupoids. Finally, we obtained necessary and sufficient conditions of a Γ -primary ideals to be a quasi Γ -primary ideals in Γ -AG-groupoids.

Keywords : Γ -AG-groupoid, Γ -LA-semigroup, Γ -ideal, quasi Γ -primary ideal, Γ -primary ideal

INTRODUCTION

Abel-Grassmann's groupoid (AG-groupoid) is the generalization of semigroup theory with wide range of usages in theory of flocks (Naseeruddin, 1970). The fundamentals of this non-associative algebraic structure were first discovered by Kazim and Naseeruddin (1972). AG-groupoid is a non-associative algebraic structure mid way between a groupoid and a commutative semigroup. It is interesting to note that an AG-groupoid with right identity becomes a commutative monoid (Mushtaq and Yousuf, 1978). This structure is closely related with a commutative semigroup because if an AG-groupoid contains a right identity, then it becomes a commutative monoid [5]. A left identity in an AG-groupoid is unique [5]. It is a mid structure between a groupoid and a commutative semigroup with wide range of applications in theory of flocks [6]. Ideals in AG-groupoids have been discussed in [5] and [4]. In 1981 the notion of Γ -semigroups was introduced by M. K. Sen [8] and [12]. A groupoid S is called a Γ -AG-groupoid if it satisfies the left invertive law:

$$(ayb)\delta c = (cyb)\delta a$$



for all $a, b, c \in S$ and $\gamma, \delta \in \Gamma$ (See [3]). This structure is also known as left almost semigroup (LA-semigroup). In this paper we are going to investigate some interesting properties of newly discovered classes of namely; Γ -AG-groupoid S always satisfies the Γ -medial law:

$$(\alpha\gamma b)\beta(c\delta d) = (\alpha\gamma c)\beta(b\delta d)$$

for all $a, b, c, d \in S$ and $\gamma, \beta, \delta \in \Gamma$ (See [3]), while a Γ -AG-groupoid S with left identity e always satisfies Γ -paramedial law:

$$(\alpha\gamma b)\beta(c\delta d) = (d\gamma c)\beta(b\delta a)$$

for all $a, b, c, d \in S$ and $\gamma, \beta, \delta \in \Gamma$ (See[3]).

In this paper we characterize the Γ -AG-groupoid. We investigate relationships between Γ -primary and quasi Γ -primary ideals in Γ -AG-groupoids.

BASIC RESULTS

In this section we refer to [11, 12] for some elementary aspects and quote few definitions and examples which are essential to step up this study. For more detail we refer to the papers in the references.

Example 2.1.[11] (1). Let S be an arbitrary AG-groupoid and Γ any non-empty set. Define a mapping $S \times \Gamma \times S \rightarrow S$; by $\alpha\gamma b = ab$ for all $a, b \in S$ and $\gamma \in \Gamma$. It is easy to see that S is a Γ -AG-groupoid.

(2). Let $\Gamma = \{1, 2, 3\}$. Define a mapping $\mathbb{Z} \times \Gamma \times \mathbb{Z} \rightarrow \mathbb{Z}$ by $\alpha\gamma b = b - \gamma - a$ for all $a, b \in \mathbb{Z}$ and $\gamma \in \Gamma$ where "-" is a usual subtraction of integers. Then $\mathbb{Z}$ is a Γ -AG-groupoid.

Lemma 2.2.[12] Every Γ -AG-groupoid is Γ -medial.

Proof. Same as [12]. □

Lemma 2.3. [10, 12] Let S be a Γ -AG-groupoid with a left identity, then $\alpha\gamma(b\alpha c) = b\gamma(a\alpha c)$ for all $a, b, c \in S$ and $\gamma, \alpha \in \Gamma$.

Proof. Same as [10, 12]. □

Definition 2.4.[11] Let S be a Γ -AG-groupoid. A nonempty subset A of S is called a sub Γ -AG-groupoid of S if $A\Gamma A \subseteq A$.

Definition 2.5. [11] A sub Γ -AG-groupoid A of S is called a left (right) Γ -ideal of S if $ST A \subseteq A$ and is called a Γ -ideal if it is left as well as right Γ -ideal.

Lemma 2.6.[11] If a Γ -AG-groupoid S has a left identity, then every right Γ -ideal is a left Γ -ideal.

Proof. Same as [11]. □



Lemma 2.7.[11] If A is a left Γ -ideal of a Γ -AG-groupoid S with left identity, and if for any $a \in S$, there exists $\gamma \in \Gamma$, then $a\gamma A$ is a left Γ -ideal of S .

Proof. Same as [11]. □

Lemma 2.11.[11] If A is a proper right(left) Γ -ideal of a Γ -AG-groupoid S with left identity e , then $e \notin A$.

Proof. Same as [11]. □

Γ -IDEALS IN Γ -AG-GROUPOIDS

The results of the following lemmas seem to be at the heart of the theory of Γ -AG-groupoids; these facts will be used so frequently that normally we shall make no reference to this lemma.

Lemma 3.1. If S is a Γ -AG-groupoid with left identity, then $a\gamma b = a\beta b$ for all $a, b \in S$ and $\gamma, \beta \in \Gamma$

Proof. Let S be a Γ -AG-groupoid and e be the left identity of S , $a, b \in S$ and let $\gamma, \beta \in \Gamma$ therefore we have

$$\begin{aligned} a\gamma b &= a\gamma(e\beta b) \\ &= e\gamma(a\beta b) \\ &= a\beta b. \end{aligned}$$

Hence $a\gamma b = a\beta b$. □

Lemma 3.2. Let S be a Γ -AG-groupoid with left identity, and let B be a left Γ -ideal of S . Then $A\Gamma B = \{a\gamma b : a \in A, b \in B, \gamma \in \Gamma\}$ is a left Γ -ideal in S , where $\emptyset \neq A \subseteq S$.

Proof. Suppose that S is a Γ -AG-groupoid with left identity. Let B be a left Γ -ideal of S . Then

$$S\Gamma(A\Gamma B) = A\Gamma(S\Gamma B) \subseteq A\Gamma B.$$

By Definition of left Γ -ideal, we get $A\Gamma B$ is a left Γ -ideal in S . □

Lemma 3.3. Let S be a Γ -AG-groupoid with left identity and let $a \in S$. Then $a^2\gamma S$ is a Γ -ideal in S , where $\gamma \in \Gamma$.

Proof. By Lemma 2.7, we have $a^2\gamma S$ is a left Γ -ideal of S . Now consider

$$\begin{aligned} (a^2\gamma r)\alpha s &= (s\gamma r)\alpha a^2 \\ &= a^2\alpha(r\gamma s) \\ &= a^2\gamma(r\gamma s) \in a^2\gamma S \end{aligned}$$

for all $r, s \in S$ and $\alpha \in \Gamma$. Therefore $a^2\gamma S$ is a Γ -ideal in S . □



Lemma 3.4. Let S be a Γ -AG-groupoid with left identity, and let A be a left Γ -ideal of S . Then $(A:\gamma:r)$ is a left Γ -ideal in S , where $(A:\gamma:r) = \{a \in S : r\gamma a \in A\}$.

Proof. Suppose that S is a Γ -AG-groupoid. Let $s \in S, a \in (A:\gamma:r)$. Then $r\gamma a \in A$ and so that

$$r\gamma(s\alpha a) = r\alpha(s\gamma a) = s\alpha(r\gamma a) \in s\alpha A \subseteq A$$

for all $\alpha \in \Gamma$. Therefore $s\alpha a \in (A:\gamma:r)$ so that $S\Gamma(A:\gamma:r) \subseteq (A:\gamma:r)$. Hence $(A:\gamma:r)$ is a left Γ -ideal in S . $\square$

Remark Let S be a Γ -AG-groupoid and let A be a left Γ -ideal of S . It is easy to verify that $A \subseteq (A:\gamma:r)$.

Lemma 3.5 Let S be a Γ -AG-groupoid with left identity, and let A, B be left ideals of S . Then $(A:\Gamma:B)$ is a Γ -left ideal in S , where $(A:\Gamma:B) = \{r \in S : B\Gamma r \subseteq A\}$.

Proof. Suppose that S is a Γ -AG-groupoid. Let $s \in S$ and let $a \in (A:\Gamma:B)$. Then $B\Gamma a \subseteq A$ so that

$$B\alpha(s\gamma a) = s\alpha(B\gamma a) \subseteq s\alpha(B\Gamma a) \subseteq s\alpha A \subseteq A$$

for all $\gamma, \beta \in \Gamma$. Therefore $s\gamma a \in (A:\Gamma:B)$ so that $S\Gamma(A:\Gamma:B) \subseteq (A:\Gamma:B)$. Hence $(A:\Gamma:B)$ is a left Γ -ideal in S . $\square$

Remark Let S be a Γ -AG-groupoid and let A, B, C be left Γ -ideals of S . It is easy to verify that $(A:\Gamma:C) \subseteq (A:\Gamma:B)$, where $B \subseteq C$.

PROPERTIES OF Γ -PRIMARY IDEALS IN Γ -AG-GROUPOIDS

We start with the following theorem that gives a relation between Γ -primary and quasi Γ -primary ideal in Γ -AG-groupoid. Our starting points is the following definition:

Definition 4.1. A Γ -ideal P is called Γ -primary if $A\Gamma B \subseteq P$ implies that $A \subseteq P$ or $B^n \subseteq P$ for some positive integer n , where A, B is a Γ -ideals of S .

Definition 4.2. A left Γ -ideal P is called quasi Γ -primary if $A\Gamma B \subseteq P$ implies that $A \subseteq P$ or $B^n \subseteq P$ for some positive integer n , where A, B is a left Γ -ideals of S .

Remark it is easy to see that every quasi Γ -primary ideal is Γ -primary.



Lemma 4.3. If S is a Γ -AG-groupoid with left identity, then a left Γ -ideal P of S is quasi Γ -primary if and only if $a\gamma(S\alpha b) \subseteq P$ implies that $a \in P$ or $b^n \in P$ for some positive integer n , where $\gamma, \beta \in \Gamma$ and $a, b \in S$.

Proof. Let P be a quasi Γ -primary left ideal of a Γ -AG-groupoid S with left identity. Now suppose that $a\gamma(S\beta b) \subseteq P$. Then by Definition of left Γ -ideal, we get $ST(a\gamma(S\beta b)) \subseteq STP \subseteq P$ that is

$$\begin{aligned} ST(a\gamma(S\beta b)) &= (S\delta S)\Gamma(a\gamma(S\beta b)) \\ &= (S\delta a)\Gamma(S\gamma(S\beta b)) \\ &= (S\delta a)\Gamma((ST\gamma)(S\beta b)) \\ &= (S\delta a)\Gamma((b\beta S)\gamma(ST)) \\ &= (S\delta a)\Gamma((b\beta S)\gamma(S)) \\ &= (S\delta a)\Gamma((S\beta S)\gamma b) \\ &= (S\delta a)\Gamma(S\gamma b) \end{aligned}$$

for all $\delta \in \Gamma$. Then $(S\delta a)\Gamma(S\gamma b) = a\gamma(S\beta b) \subseteq P$ so that $a \in \langle a \rangle \subseteq P$ or $a^n \in \langle a \rangle^n \subseteq P$, for some positive integer n . Conversely, assume that if $a\gamma(S\beta b) \subseteq P$ implies that $a \in P$ or $b^n \in P$ for some positive integer n , where $\gamma, \beta \in \Gamma$ and $a, b \in S$. Suppose that $A\Gamma B \subseteq P$, where A and B are left Γ -ideals of S such that $A \not\subseteq P$. Then there exists $x \in A$ such that $x \notin P$. Now $x\gamma(S\beta y) \subseteq A\Gamma(S\beta B) \subseteq A\Gamma B \subseteq P$, for all $y \in B$. So by hypothesis, $y^n \in P$ for all $y \in B$ implies that $B^n \subseteq P$. Hence P is quasi Γ -primary ideal in S . $\square$

Lemma 4.4. If S is a Γ -AG-groupoid with left identity, then a left Γ -ideal P of S is quasi Γ -primary if and only if $(S\gamma a)\delta(S\beta b) \subseteq P$ implies that $a \in P$ or $b^n \in P$ for some positive integer n , where $\gamma, \beta, \delta \in \Gamma$ and $a, b \in S$.

Proof. Let P be a quasi Γ -primary ideal of a Γ -AG-groupoid S with left identity. Now suppose that $(S\gamma a)\delta(S\beta b) \subseteq P$. Then by Definition of left ideal, we get

$$\begin{aligned} (S\alpha a)\delta(S\beta b) &= (S\gamma S)\delta(a\beta b) \\ &= S\delta(a\beta b) \\ &= a\delta(S\beta b) \end{aligned}$$

that is $a\delta(S\beta b) = (S\gamma a)\delta(S\beta b) \subseteq P$. By Lemma 4.3, we have $a \in P$ or $b^n \in P$ for some positive integer n . Conversely, assume that if $(S\gamma a)\delta(S\beta b) \subseteq P$, then $a \in P$ or $b^n \in P$ for some positive integer n , where $\gamma, \beta, \delta \in \Gamma$ and $a, b \in S$. Let $a\delta(S\beta b) \subseteq P$. Now consider

$$a\delta(S\beta b) = (S\gamma a)\delta(S\beta b) \subseteq P.$$



By using given assumption, if $a\delta(S\beta b) \subseteq P$, then $a \in P$ or $b^n \in P$ for some positive integer n .
Then by Lemma 4.3, we have P is a quasi Γ -primary ideal in S . $\square$

Theorem 4.5. If S is a Γ -AG-groupoid with left identity, then a left Γ -ideal P of S is quasi Γ -primary if and only if $a\gamma b \in P$ implies that $a \in P$ or $b^n \in P$ for some positive integer n , where $\gamma \in \Gamma$ and $a, b \in S$.

Proof. Let P be a left Γ -ideal of a Γ -AG-groupoid S with left identity. Now suppose that $a\gamma b \in P$. Then by Definition of left ideal, we get

$$\begin{aligned} (S\alpha a)\beta(S\gamma b) &= (S\alpha S)\beta(a\gamma b) \\ &= S\beta(a\gamma b) \\ &\subseteq S\Gamma P \\ &\subseteq P. \end{aligned}$$

By Lemma 4.4, we have $a \in P$ or $b^n \in P$ for some positive integer n .
Conversely, the proof is easy. $\square$

Theorem 4.6. Let S be a Γ -AG-groupoid, and let A be a quasi Γ -primary ideal of S . Then $(A:\gamma:r)$ is a quasi Γ -primary ideal in S , where $\gamma \in \Gamma$ and $r \in S$.

Proof. Assume that A is a quasi Γ -primary ideal of S . By Lemma 3.3, we have $(A:\gamma:r)$ is a left ideal in S . Let $a\beta b \in (A:\gamma:r)$. Suppose that $b^n \notin (A:\gamma:r)$, for all positive integer n . Since $a\beta b \in (A:\gamma:r)$ we have $r\gamma(a\beta b) \in A$ so that $a\gamma(r\beta b) \in A$. By Theorem 4.5, we have $a \in A \subseteq (A:\gamma:r)$ or $(r\beta b)^n \in A$, for some positive integer n . Therefore $a \in (A:\gamma:r)$ and hence $(A:\gamma:r)$ is a quasi Γ -primary ideal in S . $\square$

Theorem 4.7. Let S be a Γ -AG-groupoid with left identity, and let P be a Γ -primary ideal of S . If $(S\gamma a^2)\alpha(S\beta b^2) \subseteq P$, then $a^2 \in P$ or $b^n \in P$, for some positive integer n .

Proof. Let P be a Γ -primary ideal of a Γ -AG-groupoid S with left identity. Suppose that $b^n \notin P$, for all positive integer n . Now assume that $(S\gamma a^2)\alpha(S\beta b^2) \subseteq P$. Then by Definition of left ideal, we get

$$\begin{aligned} (S\gamma a^2)\alpha(S\beta b^2) &= ((S\beta b^2)\gamma a^2)\alpha S \\ &= ((a^2\beta b^2)\gamma S)\alpha S \\ &= (S\gamma S)\alpha(a^2\beta b^2) \\ &= a^2\alpha((S\gamma S)\beta b^2) \end{aligned}$$



$$= a^2 \alpha((b^2 \gamma S) \beta S)$$

$$= (b^2 \gamma S) \alpha(a^2 \beta S)$$

that is $(b^2 \gamma S) \alpha(a^2 \beta S) \subseteq P$. By Lemma 3.2, we have $a^2 \beta S, b^2 \gamma S$ are Γ -ideals in S so that

$$\begin{aligned} a^2 &= a \lambda a \\ &= (e \chi a) \lambda a \\ &= (a \chi a) \lambda e \\ &= (a \lambda a) \beta e \\ &= a^2 \beta e \in a^2 \beta S \subseteq P \end{aligned}$$

$$\begin{aligned} \text{or } b^2 &= b \lambda b \\ &= (e \chi b) \lambda b \\ &= (b \chi b) \lambda e \\ &= (b \lambda b) \gamma e \\ &= b^2 \gamma e \in b^2 \gamma S \subseteq P. \end{aligned}$$

Therefore $a^2 \in P$. □

Corollary 4.8. Let S be a Γ -AG-groupoid with left identity, and let P be a Γ -primary ideal of S . If $b^2 \gamma a^2 \in P$, then $a^2 \in P$ or $b^n \in P$, for some positive integer n .

Proof. Let P be a primary ideal of an AG-groupoid S with left identity. Suppose that $b^n \notin P$, for all positive integer n . Now assume that $b^2 \gamma a^2 \in P$. Then by Definition of left Γ -ideal, we get

$$\begin{aligned} (a^2 \beta S) \alpha(b^2 \gamma S) &= b^2 \alpha((a^2 \beta S) \gamma S) \\ &= b^2 \alpha((S \beta S) \gamma a^2) \\ &= (S \beta S) \alpha(b^2 \gamma a^2) \\ &= S \alpha(b^2 \gamma a^2) \\ &\subseteq S \Gamma P \\ &\subseteq P \end{aligned}$$

that is $(a^2 \beta S) \alpha(b^2 \gamma S) \subseteq P$. It is easy to see that $a^2 \in P$. □

Definition 4.9. A Γ -AG-groupoid S is called Γ -AG-3-band if its every element satisfies

$$a \alpha(a \beta a) = (a \alpha a) \beta a = a.$$

Proposition 4.10. Every left identity in a Γ -AG-3-band is a right identity.

Proof. Let e be a left identity and a be any element in a Γ -AG-3-band S . Then

$$\begin{aligned} aye &= (a\alpha(a\beta a))\gamma e \\ &= (e\alpha(a\beta a))\gamma a \\ &= (a\beta a)\gamma a \\ &= a \end{aligned}$$

for all $a \in S$ and $\gamma, \alpha, \beta \in \Gamma$. Hence e is right identity. $\square$

Lemma 4.11 If a Γ -AG-3-band S has a left identity, then every left Γ -ideal is a Γ -ideal.

Proof. Let S be a Γ -AG-3-band and let A be a left Γ -ideal. Then

$$\begin{aligned} ays &= ((a\alpha a)\beta a)\gamma s \\ &= (s\beta a)\gamma(a\alpha a) \\ &\in (S\Gamma A)\Gamma(A\Gamma A) \\ &\subseteq A\Gamma A \\ &\subseteq A \end{aligned}$$

for all $\gamma, \alpha, \beta \in \Gamma$, $a \in A$ and $s \in S$. Hence A is a Γ -ideal. $\square$

Theorem 4.12. Let S be a Γ -AG-3-band with left identity. Then P is a quasi Γ -primary ideal in S if and only if S is a Γ -primary ideal in S .

Proof. The proof is easy. $\square$

CONCLUSIONS

Many new classes of Γ -AG-groupoids have been discovered recently. All this has attracted researchers of the field to investigate these newly discovered classes in detail. This current article investigates the Γ -ideals, Γ -primary and quasi Γ -primary ideals in Γ -AG-groupoids. Some characterizations of Γ -primary ideal and quasi Γ -primary ideals are obtained. Moreover, we investigate relationships between Γ -primary and quasi Γ -primary ideals in Γ -AG-groupoids. Finally, we obtain necessary and sufficient conditions of a Γ -primary ideals to be a quasi Γ -primary ideals in Γ -AG-groupoids.

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