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Prime period-6 solutions of a certain piecewise linear system of difference equations

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Abstract: In this paper we consider a system of piecewise linear difference equations where the initial condition is in the closed first quadrant, the open second quadrant and the closed third quadrant. We show that there exists exactly two prime period-6 solutions, and that every solution of the system is eventually one of the two prime period-6 solutions.

Key-Words: periodic solution; systems of piecewise linear difference equations.

1 Introduction

During the last three years we have been interested in the global behavior of systems of piecewise linear difference equations. This paper is part of a general project which involves the following system

$$\begin{cases} x_{n+1} = |x_n| + ay_n + b \\ y_{n+1} = x_n + c|y_n| + d \end{cases}, \quad n = 0, 1, \dots (\mathcal{N})$$

where the parameters $a, b, c, d \in \{-1, 0, 1\}$ and initial conditions $(x_0, y_0) \in \mathbb{R}^2$. There are 81 special cases. The system's number $\mathcal{N}$ is given by

$$\mathcal{N} = 27(a+1) + 9(b+1) + 3(c+1) + (d+1) + 1.$$

This family of piecewise linear difference equations are the prototypes for more elaborate piecewise difference equations that, in many cases, exhibit complicated behavior. Interest in the area began in 1984 when Davaney published his famous paper introducing the gingerbread man map:

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1 \\ y_{n+1} = x_n \end{cases}, \quad n = 0, 1, \dots$$

with parameters $a, b \in \mathbb{R}$ and initial conditions $(x_0, y_0) \in \mathbb{R}^2$. See [1, 2].

The gingerbread man map was Devaney's response to the 1978 generalized Lozi equation:

$$\begin{cases} x_{n+1} = -a|x_n| + y_n + 1 \\ y_{n+1} = bx_n \end{cases}, \quad n = 0, 1, \dots$$

with parameters $a, b \in \mathbb{R}$ and initial conditions $(x_0, y_0) \in \mathbb{R}^2$. See [3, 4].

The Lozi equation had been used to examine an attractor that was observed by Lorenz in the Henon map, a non-linear system of difference equations

$$\begin{cases} x_{n+1} = -ax_n^2 + y_n + 1 \\ y_{n+1} = bx_n \end{cases}, \quad n = 0, 1, \dots$$

with parameters $a, b \in \mathbb{R}$ and initial conditions $(x_0, y_0) \in \mathbb{R}^2$ that modeled weather patterns. See [5].

More recently, piecewise linear difference equations have applications in neural networks and cellular neural networks, which are capable of high-speed parallel signal processing. See [6].

For other systems of this form, see [7, 8, 9, 10, 11].

2 Problem Formulation

Consider the system of piecewise linear difference equations

$$\begin{cases} x_{n+1} = |x_n| - y_n - 1 \\ y_{n+1} = x_n - |y_n| - 1 \end{cases}, \quad n = 0, 1, \dots \quad (1)$$

where the initial condition $(x_0, y_0) \in \mathbb{R}^2 - \{(x, y) : x > 0, y < 0\}$. The unique equilibrium point of System(1) is $(\bar{x}, \bar{y}) = (1, -1)$.

We show that every solution of System(1) is eventually one of the following period-6 cycles:

$$P_6^1 = \begin{pmatrix} x_0 = 3, & y_0 = -3 \\ x_1 = 5, & y_1 = -1 \\ x_2 = 5, & y_2 = 3 \\ x_3 = 1, & y_3 = 1 \\ x_4 = -1, & y_4 = -1 \\ x_5 = 1, & y_5 = -3 \end{pmatrix},$$

$$P_6^2 = \begin{pmatrix} x_0 = \frac{7}{5}, & y_0 = -3 \\ x_1 = \frac{17}{5}, & y_1 = -\frac{13}{5} \\ x_2 = 5, & y_2 = -\frac{1}{5} \\ x_3 = \frac{21}{5}, & y_3 = \frac{19}{5} \\ x_4 = -\frac{3}{5}, & y_4 = -\frac{3}{5} \\ x_5 = -\frac{1}{5}, & y_5 = \frac{11}{5} \end{pmatrix}.$$

3 Result

Set

$$Q_1 = \{(x, y) : x \geq 0, y \geq 0\}$$

$$Q_2 = \{(x, y) : x < 0, y > 0\}$$

$$Q_3 = \{(x, y) : x \leq 0, y \leq 0\}.$$

Theorem 1 Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(1) with $(x_0, y_0) \in \mathbf{R}^2 - \{(x, y) : x > 0, y < 0\}$. Then, there exists a non-negative integer $N \geq 0$ such that the solution $\{(x_n, y_n)\}_{n=N}^{\infty}$ of System(1) is either the prime period-6 cycle P_6^1 or the prime period-6 cycle P_6^2 .

The proof of Theorem 1 is a direct consequence of the following lemmas.

Lemma 2 Assume that there is a positive integer N such that $x_N = y_N \geq 0$. Then, $\{(x_n, y_n)\}_{n=N+1}^{\infty}$ is the prime period-6 cycle P_6^1 .

Proof: Suppose that (x_N, y_N) satisfies the hypothesis, then

$$x_{N+1} = x_N - x_N - 1 = -1,$$

$$y_{N+1} = x_N - x_N - 1 = -1.$$

$$\text{Hence, } (x_{N+1}, y_{N+1}) = (-1, -1) \in P_6^1. \quad \square$$

Lemma 3 Assume that there is a positive integer N such that $x_N = -y_N - 2 \leq 0$ and $y_N \leq 0$. Then, $\{(x_n, y_n)\}_{n=N+1}^{\infty}$ is the prime period-6 cycle P_6^1 .

Proof: Suppose that (x_N, y_N) satisfies the hypothesis, then

$$x_{N+1} = y_N + 2 - y_N - 1 = 1,$$

$$y_{N+1} = -y_N - 2 + y_N - 1 = -3.$$

$$\text{Hence, } (x_{N+1}, y_{N+1}) = (1, -3) \in P_6^1. \quad \square$$

Lemma 4 Let $\mathcal{S} := \{(x, -3) | x \in \mathbf{R}\}$. Then, every solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of System (1) with an initial condition in $\mathcal{S}$ is eventually a prime period-6 cycle (P_6^1 or P_6^2).

Proof: Case 1: Suppose $(x_0, y_0) \in \mathcal{S}$ and $x_0 \geq 0$. Then, $x_1 = x_0 + 2 > 0$ and $y_1 = x_0 - 4$.

If $y_1 = x_0 - 4 \geq 0$, then $(x_3, y_3) = (-1, -1) \in P_6^1$. Suppose that $y_1 = x_0 - 4 < 0$, then $x_2 = 5$ and $y_2 = 2x_0 - 3$.

Case 1.1: Suppose $y_2 = 2x_0 - 3 \geq 0 \left(\frac{3}{2} \leq x_0 < 4\right)$, then $x_3 = -2x_0 + 7$ and $y_3 = -2x_0 + 7$.

If $x_3 = y_3 = -2x_0 + 7 \geq 0 \left(\frac{3}{2} \leq x_0 \leq \frac{7}{2}\right)$, then by Lemma 2, $\{(x_n, y_n)\}_{n=3}^{\infty}$ is eventually prime period-6 (P_6^1).

Suppose that $x_3 = y_3 = -2x_0 + 7 < 0 \left(\frac{7}{2} < x_0 < 4\right)$. We will prove, by mathematical induction, that for $x_0 \in \left(\frac{7}{2}, 4\right)$ the solution is eventually prime period-6.

For each $n \geq 0$, let $P(n)$ be the following statement: for $x_0 \in (a_n, b_n)$,

$$x_{6n+4} = 2^{4n+2}x_0 - \delta_n,$$

$$y_{6n+4} = -2^{4n+2}x_0 + \delta_n - 2 < 0.$$

If $x_0 \in (a_n, c_n]$, then $x_{6n+4} \leq 0$. So, the solution is eventually prime period-6.

If $x_0 \in (c_n, b_n)$, then $x_{6n+4} > 0$. So,

$$x_{6n+5} = 2^{4n+3}x_0 - 2\delta_n + 1 > 0,$$

$$y_{6n+5} = -3,$$

$$x_{6n+6} = 2^{4n+3}x_0 - 2\delta_n + 3 > 0,$$

$$y_{6n+6} = 2^{4n+3}x_0 - 2\delta_n - 3 < 0,$$

$$x_{6n+7} = 5,$$

$$y_{6n+7} = 2^{4n+4}x_0 - 4\delta_n - 1.$$

If $x_0 \in [b_{n+1}, b_n)$, then $y_{6n+7} \geq 0$. So, the solution is eventually prime period-6 (P_6^1).

If $x_0 \in (c_n, b_{n+1})$, then $y_{6n+7} < 0$. So

$$x_{6n+8} = -2^{4n+4}x_0 + 4\delta_n + 5 > 0,$$

$$y_{6n+8} = 2^{4n+4}x_0 - 4\delta_n + 3 > 0,$$

$$x_{6n+9} = -2^{4n+5}x_0 + 8\delta_n + 1,$$

$$y_{6n+9} = -2^{4n+5}x_0 + 8\delta_n + 1.$$

If $x_0 \in (c_n, a_{n+1}]$, then $x_{6n+9} = y_{6n+9} \geq 0$. So, $\{(x_n, y_n)\}_{n=0}^\infty$ is eventually prime period-6 (P_6^1).

If $x_0 \in (a_{n+1}, b_{n+1})$, then $x_{6n+9} = y_{6n+9} = -2^{4n+5}x_0 + 8\delta_n + 1 < 0$, where

$$\begin{aligned} a_n &= \frac{19 \times 2^{4n+1} - 3}{5 \times 2^{4n+1}}, b_n = \frac{19 \times 2^{4n} + 1}{5 \times 2^{4n}}, \\ c_n &= \frac{19 \times 2^{4n+2} - 1}{5 \times 2^{4n+2}}, \delta_n = \frac{19 \times 2^{4n+2} - 1}{5}. \end{aligned}$$

We shall show that $P(0)$ is true. For

$$x_0 \in (a_0, b_0) = \left(\frac{7}{2}, 4\right) \text{ and}$$

$$x_3 = y_3 = -2x_0 + 7 < 0. \text{ Thus, we have}$$

$$x_{6(0)+4} = x_4 = 4x_0 - 15 = 2^{4(0)+2}x_0 - \delta_0,$$

$$y_{6(0)+4} = y_4 = -4x_0 + 13 = -2^{4(0)+2}x_0 + \delta_0 - 2 < 0.$$

If $x_0 \in (a_0, c_0] = \left(\frac{7}{2}, \frac{15}{4}\right]$, then $x_4 = 4x_0 - 15 \leq 0$. By Lemma 3, $\{(x_n, y_n)\}_{n=4}^\infty$ is eventually prime period-6 (P_6^1).

If $x_0 \in (c_0, b_0) = \left(\frac{15}{4}, 4\right)$, then $x_4 = 4x_0 - 15 > 0$. Thus, we have

$$x_{6(0)+5} = 8x_0 - 29 = 2^{4(0)+3}x_0 - 2\delta_0 + 1 > 0,$$

$$y_{6(0)+5} = -3,$$

$$x_{6(0)+6} = 8x_0 - 27 = 2^{4(0)+3}x_0 - 2\delta_0 + 3 > 0,$$

$$y_{6(0)+6} = 8x_0 - 33 = 2^{4(0)+3}x_0 - 2\delta_0 - 3 < 0,$$

$$x_{6(0)+7} = 5,$$

$$y_{6(0)+7} = 16x_0 - 61 = 2^{4(0)+4}x_0 - 4\delta_0 - 1.$$

If $x_0 \in [b_1, b_0) = \left[\frac{61}{16}, 4\right)$, then $y_{6(0)+7} = 16x_0 - 61 \geq 0$. So $x_{6(0)+8} = y_{6(0)+8} = -16x_0 + 65 > 0$. By Lemma 2, $\{(x_n, y_n)\}_{n=9}^\infty$ is prime period-6.

If $x_0 \in (c_0, b_1) = \left(\frac{15}{4}, \frac{61}{16}\right)$, then $x_{6(0)+7} = 16x_0 - 61 < 0$. Thus, we have $x_{6(0)+8} = -16x_0 + 65 = -2^{4(0)+4}x_0 + 4\delta_0 + 5 > 0$, $y_{6(0)+8} = 16x_0 - 57 = 2^{4(0)+4}x_0 - 4\delta_0 + 3 > 0$, $x_{6(0)+9} = -32x_0 + 121 = -2^{4(0)+5}x_0 + 8\delta_0 + 1$, $y_{6(0)+9} = -32x_0 + 121 = -2^{4(0)+5}x_0 + 8\delta_0 + 1$.

If $x_0 \in (c_0, a_1] = \left(\frac{15}{4}, \frac{121}{32}\right]$, then $x_{6(0)+9} = y_{6(0)+9} = -32x_0 + 121 \geq 0$. So, by Lemma 2, $\{(x_n, y_n)\}_{n=10}^\infty$ is eventually prime period-6 (P_6^1).

If $x_0 \in (a_1, b_1) = \left(\frac{121}{32}, \frac{61}{16}\right)$, then $x_{6(0)+9} = y_{6(0)+9} = -32x_0 + 121 < 0$. Hence $P(0)$ is true.

Next, we assume that $P(N)$ is true for some positive integer $N \geq 1$. We shall show that $P(N+1)$ is true.

Since $P(N)$ is true, $x_{6N+9} = y_{6N+9} = -2^{4N+5}x_0 + 8\delta_N + 1 < 0$, where $x_0 \in (a_{N+1}, b_{N+1}) = \left(\frac{19 \times 2^{4N+5} - 3}{5 \times 2^{4N+5}}, \frac{19 \times 2^{4N+4} + 1}{5 \times 2^{4N+4}}\right)$. Then

$$x_{6(N+1)+4} = x_{6N+10} = 2^{4N+6}x_0 - 16\delta_N - 3$$

$$= 2^{4(N+1)+2}x_0 - \delta_{N+1},$$

$$y_{6(N+1)+4} = y_{6N+10} = -2^{4N+6}x_0 + 16\delta_N + 1$$

$$= -2^{4(N+1)+2}x_0 + \delta_{N+1} - 2$$

$$= -2^{4N+6}x_0 + \left(\frac{19 \times 2^{4N+6} - 11}{5}\right) < 0.$$

Note that

$$\delta_{N+1} = \frac{19 \times 2^{4N+6} - 1}{5} = \frac{19 \times 2^{4N+6} - 16}{5} + \frac{15}{5} = 16\delta_N + 3.$$

If

$$x_0 \in (a_{N+1}, c_{N+1}] = \left(\frac{19 \times 2^{4N+5} - 3}{5 \times 2^{4N+5}}, \frac{19 \times 2^{4N+6} - 1}{5 \times 2^{4N+6}}\right],$$

then

$$x_{6N+10} = 2^{4N+6}x_0 - \left(\frac{19 \times 2^{4N+6} - 1}{5}\right) \leq 0.$$

By Lemma 3, $\{(x_n, y_n)\}_{n=6N+10}^\infty$ is eventually prime period-6 (P_6^1).

If

$$x_0 \in (c_{N+1}, b_{N+1}) = \left(\frac{19 \times 2^{4N+6} - 1}{5 \times 2^{4N+6}}, \frac{19 \times 2^{4N+4} + 1}{5 \times 2^{4N+4}}\right),$$

then

$$x_{6N+10} = 2^{4N+6}x_0 - \left(\frac{19 \times 2^{4N+6} - 1}{5}\right) > 0.$$

Thus, we have

$$\begin{aligned} x_{6(N+1)+5} &= x_{6N+11} \\ &= 2^{4(N+1)+3}x_0 - 2\delta_{N+1} + 1 > 0, \end{aligned}$$

$$y_{6(N+1)+5} = y_{6N+11} = -3,$$

$$\begin{aligned} x_{6(N+1)+6} &= x_{6N+12} \\ &= 2^{4(N+1)+3}x_0 - 2\delta_{N+1} + 3 > 0, \end{aligned}$$

$$\begin{aligned} y_{6(N+1)+6} &= y_{6N+12} \\ &= 2^{4(N+1)+3}x_0 - 2\delta_{N+1} - 3 \\ &= 2^{4N+7}x_0 - \left(\frac{19 \times 2^{4N+7} + 13}{5}\right) < 0, \end{aligned}$$

$$x_{6(N+1)+7} = x_{6N+13} = 5,$$

$$\begin{aligned} y_{6(N+1)+7} &= y_{6N+13} \\ &= 2^{4(N+1)+4}x_0 - 4\delta_{N+1} - 1. \end{aligned}$$

If

$$x_0 \in [b_{N+2}, b_{N+1}) = \left[\frac{19 \times 2^{4N+8} + 1}{5 \times 2^{4N+8}}, \frac{19 \times 2^{4N+4} + 1}{5 \times 2^{4N+4}}\right),$$

then

$$y_{6N+13} = 2^{4(N+1)+4}x_0 - 4\delta_{N+1} - 1$$

$$\begin{aligned}
&= 2^{4N+8}x_0 - \left(\frac{19 \times 2^{4N+8} + 1}{5} \right) \geq 0, \\
x_{6N+14} &= -2^{4N+8}x_0 + 4\delta_{N+1} + 5 \\
&= -2^{4N+8}x_0 + \left(\frac{19 \times 2^{4N+8} + 21}{5} \right) > 0 \\
y_{6N+14} &= -2^{4N+8}x_0 + 4\delta_{N+1} + 5 > 0. \text{ By Lemma} \\
&2, \{(x_n, y_n)\}_{n=6N+14}^\infty \text{ is eventually prime period-6.}
\end{aligned}$$

If

$$\begin{aligned}
x_0 &\in (c_{N+1}, b_{N+2}) = \\
&\left(\frac{19 \times 2^{4N+6} - 1}{5 \times 2^{4N+6}}, \frac{19 \times 2^{4N+8} + 1}{5 \times 2^{4N+8}} \right), \\
\text{then} \\
y_{6N+13} &= 2^{4(N+1)+4}x_0 - 4\delta_{N+1} - 1 \\
&= 2^{4N+8}x_0 - \left(\frac{19 \times 2^{4N+8} + 1}{5} \right) < 0.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
x_{6(N+1)+8} &= x_{6N+14} \\
&= -2^{4(N+1)+4}x_0 + 4\delta_{N+1} + 5 > 0, \\
y_{6(N+1)+8} &= y_{6N+14} \\
&= 2^{4(N+1)+4}x_0 - 4\delta_{N+1} + 3 \\
&= 2^{4N+8}x_0 - \left(\frac{19 \times 2^{4N+8} - 19}{5} \right) > 0, \\
x_{6(N+1)+9} &= x_{6N+15} = -2^{4(N+1)+5}x_0 + 8\delta_{N+1} + 1, \\
y_{6(N+1)+9} &= y_{6N+15} = -2^{4(N+1)+5}x_0 + 8\delta_{N+1} + 1.
\end{aligned}$$

If

$$\begin{aligned}
x_0 &\in (c_{N+1}, a_{N+2}] = \\
&\left(\frac{19 \times 2^{4N+6} - 1}{5 \times 2^{4N+6}}, \frac{19 \times 2^{4N+9} - 3}{5 \times 2^{4N+9}} \right], \\
\text{then} \\
x_{6N+15} &= y_{6N+15} \\
&= -2^{4(N+1)+5}x_0 + 8\delta_{N+1} + 1 \\
&= -2^{4N+9}x_0 + \left(\frac{19 \times 2^{4N+9} - 3}{5} \right) \geq 0.
\end{aligned}$$

By Lemma 2, $\{(x_n, y_n)\}_{n=6N+15}^\infty$ is eventually prime period-6 (P_6^1).

If

$$\begin{aligned}
x_0 &\in (a_{N+2}, b_{N+2}) = \\
&\left(\frac{19 \times 2^{4N+9} - 3}{5 \times 2^{4N+9}}, \frac{19 \times 2^{4N+8} + 1}{5 \times 2^{4N+8}} \right), \\
\text{then} \\
x_{6N+15} &= y_{6N+15} \\
&= -2^{4(N+1)+5}x_0 + 8\delta_{N+1} + 1 \\
&= -2^{4N+9}x_0 + \left(\frac{19 \times 2^{4N+9} - 3}{5} \right) < 0.
\end{aligned}$$

Hence, $P(N+1)$ is true. By mathematical induction, $P(n)$ is true for all $n \geq 0$.

Note that $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} c_n = \frac{19}{5}$.

We also note that if $(x_0, y_0) = \left(\frac{19}{5}, -3 \right)$, then

$$(x_3, y_3) = \left(-\frac{3}{5}, -\frac{3}{5} \right) \in P_6^2.$$

Case 1.2: Suppose $y_2 = 2x_0 - 3 < 0$ ($0 \leq x_0 < \frac{3}{2}$), then $x_4 = -4x_0 + 5$ and $y_4 = -4x_0 + 5$.

If $x_0 \in \left(0, \frac{5}{4} \right]$, then $x_4 = y_4 = -4x_0 + 5 \geq 0$.

So we apply Lemma 2 and $\{(x_n, y_n)\}_{n=4}^\infty$ is eventually prime period-6 (P_6^1).

Suppose that $x_0 \in \left(\frac{5}{4}, \frac{3}{2} \right)$. We will prove,

for $x_0 \in \left(\frac{5}{4}, \frac{3}{2} \right)$, that the solution is eventually prime period-6 by mathematical induction. For each $n \geq 0$, let $Q(n)$ be the following statement: for $x_0 \in (d_n, e_n)$,

$$\begin{aligned}
x_{6n+5} &= 2^{4n+3}x_0 - \sigma_n, \\
y_{6n+5} &= -2^{4n+3}x_0 + \sigma_n - 2 < 0.
\end{aligned}$$

If $x_0 \in (d_n, f_n]$, then $x_{6n+5} \leq 0$. So, $\{(x_n, y_n)\}_{n=0}^\infty$ is eventually prime period-6 (P_6^1).

If $x_0 \in (f_n, e_n)$, then $x_{6n+5} > 0$. So

$$\begin{aligned}
x_{6n+6} &= 2^{4n+4}x_0 - 2\sigma_n + 1 > 0, \\
y_{6n+6} &= -3, \\
x_{6n+7} &= 2^{4n+4}x_0 - 2\sigma_n + 3 > 0, \\
y_{6n+7} &= 2^{4n+4}x_0 - 2\sigma_n - 3 < 0, \\
x_{6n+8} &= 5, \\
y_{6n+8} &= 2^{4n+5}x_0 - 4\sigma_n - 1.
\end{aligned}$$

If $x_0 \in [e_{n+1}, e_n)$, then $y_{6n+8} \geq 0$. So, the solution is eventually prime period-6 (P_6^1).

If $x_0 \in (f_n, e_{n+1})$, then $y_{6n+8} < 0$. Thus, we have

$$\begin{aligned}
x_{6n+9} &= -2^{4n+5}x_0 + 4\sigma_n + 5 > 0, \\
y_{6n+9} &= 2^{4n+5}x_0 - 4\sigma_n + 3 > 0, \\
x_{6n+10} &= -2^{4n+6}x_0 + 8\sigma_n + 1, \\
y_{6n+10} &= -2^{4n+6}x_0 + 8\sigma_n + 1.
\end{aligned}$$

If $x_0 \in (f_n, d_{n+1}]$, then $x_{6n+10} = y_{6n+10} \geq 0$, and so the solution is eventually prime period-6 (P_6^1).

If $x_0 \in (d_{n+1}, e_{n+1})$, then $x_{6n+10} = y_{6n+10} < 0$, where

$$\begin{aligned}
d_n &= \frac{7 \times 2^{4n+2} - 3}{5 \times 2^{4n+2}}, e_n = \frac{7 \times 2^{4n+1} + 1}{5 \times 2^{4n+1}}, \\
f_n &= \frac{7 \times 2^{4n+3} - 1}{5 \times 2^{4n+3}}, \sigma_n = \frac{7 \times 2^{4n+3} - 1}{5}.
\end{aligned}$$

The proof is similar to the previous case. We can conclude that $Q(n)$ is true for all $n \geq 0$.

Note that $\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} e_n = \lim_{n \rightarrow \infty} f_n = \frac{7}{5}$.

We also note that $\left(\frac{7}{5}, -3 \right) \in P_6^2$.

Case 2: Let $(x_0, y_0) \in \mathcal{S}$ and $x_0 < 0$. Then, $y_0 = -3$. Thus, $x_2 = -2x_0 + 5 > 0$ and $y_2 = -3$.

We see that $(x_2, y_2) \in \mathcal{S}$ and $x_2 > 0$. By Case 1, every solution $\{(x_n, y_n)\}_{n=2}^\infty$ is eventually prime period-6 (P_6^1 or P_6^2). $\square$

Lemma 5 Every solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of System (1) with an initial condition in Q_1 is eventually prime period-6 (P_6^1 or P_6^2).

Proof: Let $(x_0, y_0) \in Q_1$. Then $x_0 \geq 0$ and $y_0 \geq 0$. Hence,

$$x_1 = x_0 - y_0 - 1,$$

$$y_1 = x_0 - y_0 - 1.$$

If $x_1 = y_1 = x_0 - y_0 + 1 \geq 0$, then by Lemma 2, $(x_2, y_2) \in P_6^1$.

If $x_1 = y_1 = x_0 - y_0 + 1 < 0$, then

$$x_2 = -2x_0 + 2y_0 + 1,$$

$$y_2 = 2x_0 - 2y_0 - 3 < 0.$$

If $x_2 = -2x_0 + 2y_0 + 1 = -y_2 - 2 \leq 0$, then by Lemma 3, $(x_2, y_2) \in P_6^1$.

If $x_2 = -2x_0 + 2y_0 + 1 > 0$, then

$$x_3 = -4x_0 + 4y_0 + 3,$$

$y_3 = -3$. Hence by Lemma 4, the solution is eventually prime period-6 (P_6^1 or P_6^2). $\square$

Lemma 6 Every solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of System (1) with an initial condition in Q_3 is eventually prime period-6 (P_6^1 or P_6^2).

Proof: Let $(x_0, y_0) \in Q_3$. Then $x_0 \leq 0$ and $y_0 \leq 0$. Hence,

$$x_1 = -x_0 - y_0 - 1,$$

$$y_1 = x_0 + y_0 - 1 < 0,$$

$$y_2 = -3.$$

By Lemma 4, the solution is eventually prime period-6 (P_6^1 or P_6^2). $\square$

Lemma 7 Every solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of System (1) with an initial condition in Q_2 is eventually prime period-6 (P_6^1 or P_6^2).

Proof: Let $(x_0, y_0) \in Q_2$. Then $x_0 < 0$ and $y_0 > 0$. Hence,

$$x_1 = -x_0 - y_0 - 1,$$

$$y_1 = x_0 - y_0 - 1 < 0.$$

Case 1: Suppose $x_1 = -x_0 - y_0 - 1 \leq 0$. Then,

$$x_2 = 2y_0 + 1 > 0,$$

$$y_2 = -2y_0 - 3 < 0,$$

$$y_3 = -3.$$

By Lemma 4, the solution is eventually prime period-6 (P_6^1 or P_6^2).

Case 2: Suppose $x_1 = -x_0 - y_0 - 1 > 0$ and so $-x_0 > 1$. Then,

$$x_2 = -2x_0 - 1 > 0,$$

$$y_2 = -2y_0 - 3 < 0,$$

$$x_3 = -2x_0 + 2y_0 + 1 > 0,$$

$$y_3 = -2x_0 - 2y_0 - 5.$$

If $y_3 = -2x_0 - 2y_0 - 5 \geq 0$, then

$$x_4 = 4y_0 + 5 > 0,$$

$$y_4 = 4y_0 + 5 > 0.$$

By Lemma 2, the solution is eventually prime period-6 (P_6^1).

If $y_3 = -2x_0 - 2y_0 - 5 < 0$, then

$$x_4 = 4y_0 + 5 > 0,$$

$$y_4 = -4x_0 - 5.$$

If $y_4 = -4x_0 - 5 \leq 0$ $\left(-\frac{5}{4} \leq x_0 < -1\right)$, then

$$x_5 = 4x_0 + 4y_0 + 9 > 0,$$

$$y_5 = -4x_0 + 4y_0 - 1 > 0.$$

By Lemma 5, the solution is eventually prime period-6 (P_6^1 or P_6^2).

If $y_4 = -4x_0 - 5 > 0$ $\left(x_0 < -\frac{5}{4}\right)$, then

$$x_5 = 4x_0 + 4y_0 + 9,$$

$$y_5 = 4x_0 + 4y_0 + 9.$$

If $x_5 = y_5 \geq 0$, then by Lemma 5, the solution is eventually prime period-6 (P_6^1 or P_6^2).

If $x_5 = y_5 < 0$, then by Lemma 6, the solution is eventually prime period-6 (P_6^1 or P_6^2). $\square$

4 Conclusion and Discussion

We have presented the prime period-6 character of the solutions of System(1). We utilized mathematical induction, and direct computations to show that every solution of System(1) with an initial condition in $Q_1 \cup Q_2 \cup Q_3$ is eventually either the prime period-6 solution P_6^1 , or the prime period-6 solution P_6^2 . The proofs involve careful consideration of the various cases. We conjecture that every solution of System(1) with an initial condition in R^2 is eventually either the prime period-6 solution P_6^1 , the prime period-6 solution P_6^2 or the unique equilibrium point, $(1, -1)$, from the beginning.

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